A converse to equation (13.2). We saw on Thursday that the Riemann hypothesis implies that $\psi(x) = x + O(\sqrt{x}\log^2 x)$. (And by the usual methods, equally strong error terms hold for $\theta(x)$ and $\pi(x)$; see Theorem 13.1.) We actually have the tools to prove a converse of this result:

Proposition 1. *If $\psi(x) = x + O_\varepsilon(x^{1/2+\varepsilon})$ for all $\varepsilon > 0$, then the Riemann hypothesis holds.*

Proof. Consider the Dirichlet series

$$\alpha(s) = \sum_{n=1}^{\infty} \big(\Lambda(n) - 1\big)n^{-s} = -\frac{\zeta'}{\zeta}(s) - \zeta(s).$$

In the notation of Section 1.2 we have $A(x) = \sum_{n\le x} \big(\Lambda(n) - 1\big) = \psi(x) - x + O(1) \ll_\varepsilon x^{1/2+\varepsilon}$ by assumption. Theorem 1.3 then tells us that

$$\sigma_c = \limsup_{x\to\infty} \frac{\log|A(x)|}{\log x} \le \frac{1}{2} + \varepsilon,$$

and therefore the Dirichlet series $\alpha(s)$ converges for $\sigma > \frac{1}{2} + \varepsilon$; since $\varepsilon > 0$ was arbitrary, we conclude that $\alpha(s)$ converges (hence is analytic) for $\sigma > \frac{1}{2}$. But this means that $-\frac{\zeta'}{\zeta}(s) - \zeta(s)$ is analytic for $\sigma > \frac{1}{2}$, which implies that $\zeta(s)$ does not vanish in that half-plane, which is the Riemann hypothesis. □

(If we look at these two implications closely, we see that $\psi(x) = x + O_\varepsilon(x^{1/2+\varepsilon})$ implies RH which in turn implies $\psi(x) = x + O(\sqrt{x}\log^2 x)$. It's rare for such a two-way street to actually *improve* the original result!)

One can be more precise about how the error term depends upon the locations of the zeros of $\zeta(s)$; see Theorem 15.3 and Corollary 15.4. (The remark after the latter is the tiny opening that essentially leads to the entire field of "comparative prime number theory".)

(continued on next page)

Counting nontrivial zeros of $\zeta(s)$. As we did on Tuesday, define

$$N(T) = \#\big\{\rho = \beta + i\gamma \in \mathbb{C} \colon 0 < \sigma < 1,\, 0 \le \gamma \le T\big\}$$

to be the counting function of the nontrivial zeros of $\zeta(s)$ in the upper half-plane, ordered by imaginary part. (We make the usual conventions that $N(T)$ counts zeros according to their multiplicities, and that $N(T) = \frac{1}{2}\big(N(T+) + N(T-)\big)$ at its discontinuities.) This is precisely the same as the counting function of the zeros of $\xi(s)$.

In complex analysis we learned the "argument principle", that a suitable contour integral can count the number of zeros of an analytic function (indeed it works even for meromorphic functions). In particular, if we let C be the rectangle with corners at 2, $2 + iT$, $-1 + iT$, and -1, then

$$N(T) = \frac{1}{2\pi i} \oint_C -\frac{\xi'}{\xi}(s)\, ds.$$

It turns out that we can actually evaluate this contour integral, and hence $N(T)$, *exactly*, in terms of simpler functions, thanks to the functional equation.

Please read Theorem 14.1 and its proof for the exact statement.

Note: we make the following convention for the argument (and logarithm) of values of analytic functions connected to $\zeta(s)$. We declare that the argument of $\zeta(2)$ to be 0 (instead of a random multiple of 2π); and then to calculate the argument of $\zeta(\sigma + it)$, we continuously extend the argument of $\zeta(s)$ from $s = 2$ vertically to $s = 2 + it$ and then horizontally to $s = \sigma + it$.