Last Thursday we learned what a *Dirichlet character* $\chi \pmod q$ is (for any $q \in \mathbb{N}$): a totally multiplicative function, periodic with period q, supported on the integers relatively prime to q. We learned that there are $\phi(q)$ Dirichlet characters $\pmod q$, and that one of them is always the *principal character*

$$\chi_0(n) = \begin{cases} 1, & \text{if } (n,q) = 1, \\ 0, & \text{if } (n,q) > 1. \end{cases}$$

(We'll see that χ_0 behaves a bit differently from other Dirichlet characters.) We also learned that all nonzero values of a Dirichlet character $\chi \pmod q$ are $\phi(q)$th roots of unity; in particular, $|\chi(n)| \le 1$ for all $n \in \mathbb{Z}$.

Dirichlet L-functions. Just as we can for any arithmetic function, we can use a Dirichlet character to define a Dirichlet series

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

which we call a *Dirichlet L-function*. (It's no mystery which historical figure was influential to this part of the subject....) Notice that when $\sigma > 1$,

$$|L(s,\chi)| \le \sum_{n=1}^{\infty} |\chi(n)| n^{-\sigma} \le \sum_{n=1}^{\infty} 1 \cdot n^{-\sigma} < \infty;$$

therefore the Dirichlet series defining $L(s,\chi)$ converges absolutely for $\sigma > 1$ (in other words, the abscissa of absolute convergence satisfies $\sigma_a \le 1$).

Exercise: show that the series defining $L(1,\chi)$ does not converge absolutely, and that the series defining $L(1,\chi_0)$ does not converge at all. (Hint: consider just the summands $n \equiv 1 \pmod q$.) Conclude that $\sigma_a = 1$ for any Dirichlet L-function and that $\sigma_c = 1$ for the Dirichlet L-function of a principal character.

Since $\chi(n)$ is totally multiplicative, when $\sigma > 1$ we therefore have the Euler product

$$L(s,\chi) = \prod_p \left(1 + \frac{\chi(p)}{p^{-s}} + \frac{\chi(p)^2}{p^{-2s}} + \cdots \right) = \prod_p \left(1 - \frac{\chi(p)}{p^{-s}}\right)^{-1}. \tag{1}$$

Exercise: if χ_0 is the principal character $\pmod q$, show that when $\sigma > 1$,

$$L(s,\chi_0) = \zeta(s) \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right),$$

Conclude that $L(s,\chi_0)$ has a meromorphic continuation to $\sigma > 0$, with its only pole being a simple pole at $s = 1$ with residue $\phi(q)/q$.

Abscissa of convergence. On Thursday we also learned two orthogonality relations for Dirichlet characters, one of which was equation (4.14):

$$\sum_{n=1}^{q} \chi(n) = \begin{cases} \phi(q), & \text{if } \chi = \chi_0, \\ 0, & \text{if } \chi \ne \chi_0. \end{cases}$$

(The statement in the book has the additional restriction $(n, q) = 1$, but of course this can be removed since the additional summands all equal 0.) It easily follows from periodicity that when χ is nonprincipal, the sum of $\chi(n)$ over any interval whose length is a multiple of q equals 0. Therefore, if we define $A_\chi(x) = \sum_{n\le x} \chi(n)$, it follows that $A_\chi(x)$ is uniformly bounded when χ is nonprincipal: if we write $x = qv + w$ with $0 \le w < q$, then

$$|A_\chi(x)| = \Big|\sum_{n\le x} \chi(n)\Big| = \Big|\sum_{n\le w} \chi(n) + \sum_{w<n\le qv+w} \chi(n)\Big| = \Big|\sum_{n\le w} \chi(n) + 0\Big| \le \sum_{n\le w} |\chi(n)| \le \sum_{n\le w} 1 < q.$$

Exercise: show that in fact $|A_\chi(x)| \le \phi(q)/2$ when χ is nonprincipal.

This bound has a nice implication when combined with Theorem 1.3: when χ is nonprincipal,

$$\sigma_c = \limsup_{x\to\infty} \frac{\log |A_\chi(x)|}{\log x} \le \limsup_{x\to\infty} \frac{\log q}{\log x} = 0.$$

(Theorem 1.3 assumes that $\sigma_c \ge 0$, but it's easy to see that the series defining $L(s, \chi)$ does not converge when $\sigma = 0$, as the summand does not tend to 0.)

In other words, the Dirichlet series defining $L(s, \chi)$ actually converges for all $\sigma > 0$ when $\chi \ne \chi_0$. (Note, however, that the Euler product (1) does not necessarily converge in this larger half plane, since Theorem 1.9 requires absolute convergence.)

You can now read the beginning of Section 4.3, through Theorem 4.8 and the following paragraph.

Nonvanishing of $L(1, \chi)$. It turns out to be very important that $L(1, \chi)$ is never equal to 0. The standard proofs of this statement partition the set of all Dirichlet characters into three types:

- χ is *principal* (a definition we've already seen, $\chi = \chi_0$);
- χ is *quadratic*, meaning that $\chi^2 = \chi_0$ but $\chi \ne \chi_0$;
- χ is *complex*, meaning that the values of χ are not all real.

Indeed, the principal and quadratic characters together form the *real* characters, meaning that the values of χ are all real (confirm this from the definitions above). For the present purposes, it suffices to consider nonprincipal χ, since $L(s, \chi)$ has a pole at $s = 1$ and hence certainly does not equal 0.

Here is a sketch of a proof, for complex characters, that in fact $L(1 + it, \chi) \ne 0$ for all $t \in \mathbb{R}$, and in particular that $L(1, \chi) \ne 0$. When $\sigma > 1$, the unexpected inequality

$$\big|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)\big| \ge 1$$

that we proved earlier in the course can be easily generalized (*exercise*) to

$$\big|L(\sigma, \chi_0)^3 L(\sigma + it, \chi)^4 L(\sigma + 2it, \chi^2)\big| \ge 1. \tag{2}$$

If $L(1 + it) = 0$, then the function $L(s, \chi_0)^3 L(s + it, \chi)^4 L(s + 2it, \chi^2)$ would have a triple pole times (at least) a quadruple zero, hence would vanish at $s = 1$, contradicting the inequality (2) as $\sigma \to 1+$. (This deduction requires $L(s + 2it)$ to be analytic near $s = 1$, which holds whenever $t \ne 0$ or whenever $\chi^2 \ne \chi_0$; this is where we use the assumption that χ is complex.) This is essentially the same proof that $\zeta(s)$ does not vanish when $\sigma = 1$.

For quadratic characters, however, we must find another proof; here is a sketch of the one in the book. If we define $r = \chi * 1$, then one can show that r is a nonnegative multiplicative function and that $r(n^2) \ge 1$ for all $n \in \mathbb{N}$. But $\sum_{n=1}^{\infty} r(n)n^{-s} = L(s,\chi)\zeta(s)$. If $L(1,\chi) = 0$, then this product would be analytic at $s = 1$; from this one can deduce a contradiction from Landau's theorem.

At this point, you can read the proof of Theorem 4.9, found on pages 123–124, a bit separated from the statement of the theorem. Note that the book gives a different, equally interesting proof of $L(1,\chi) \ne 0$ *when* χ *is complex.*

Remark: it turns out that there is a "zero-free region" for these Dirichlet L-functions as well; however, the distinction between complex and quadratic characters grows even more stark when establishing such a region, and the result for quadratic characters is still somewhat unsatisfactory. We would cover such a result in the topics course I'll teach in the fall, assuming that it is offered (as I hope).

Applications to primes in arithmetic progressions. Last Thursday we also learned a second orthogonality relation, namely equation (4.15):

$$\frac{1}{\phi(q)} \sum_{\chi \ (\mathrm{mod}\ q)} \chi(n) = \begin{cases} 1, & \text{if } n \equiv 1 \ (\mathrm{mod}\ q), \\ 0, & \text{if } n \not\equiv 1 \ (\mathrm{mod}\ q). \end{cases}$$

(Recall that $\sum_{\chi \ (\mathrm{mod}\ q)}$ denotes a sum over all $\phi(q)$ Dirichlet characters (mod q). The fact that 1 (mod q) is special in this relation, while χ_0 is special in the other orthogonality relation, is due to the fact that both are the identity elements in their respective groups.) The change of variables $n \mapsto a^{-1}n$ (mod q) then shows that

$$\begin{aligned} \frac{1}{\phi(q)} \sum_{\chi \ (\mathrm{mod}\ q)} \overline{\chi(a)}\chi(n) &= \frac{1}{\phi(q)} \sum_{\chi \ (\mathrm{mod}\ q)} \chi(a^{-1}n) \\ &= \begin{cases} 1, & \text{if } a^{-1}n \equiv 1 \ (\mathrm{mod}\ q), \\ 0, & \text{if } a^{-1}n \not\equiv 1 \ (\mathrm{mod}\ q) \end{cases} = \begin{cases} 1, & \text{if } n \equiv a \ (\mathrm{mod}\ q), \\ 0, & \text{if } n \not\equiv a \ (\mathrm{mod}\ q). \end{cases} \end{aligned}$$

We can therefore use these Dirichlet characters to isolate a particular reduced residue class a (mod q):

$$\begin{aligned} \sum_{\substack{n\in\mathbb{N} \\ n\equiv a \ (\mathrm{mod}\ q)}} \Lambda(n)n^{-s} &= \sum_{n\in\mathbb{N}} \Lambda(n)n^{-s} \cdot \frac{1}{\phi(q)} \sum_{\chi \ (\mathrm{mod}\ q)} \overline{\chi(a)}\chi(n) \\ &= \frac{1}{\phi(q)} \sum_{\chi \ (\mathrm{mod}\ q)} \overline{\chi(a)} \sum_{n\in\mathbb{N}} \Lambda(n)\chi(n)n^{-s} = -\frac{1}{\phi(q)} \sum_{\chi \ (\mathrm{mod}\ q)} \overline{\chi(a)} \frac{L'}{L}(s,\chi), \end{aligned}$$

where the last equality used equation (4.25) that you read about earlier today. (An analogous computation would work for any arithmetic function in place of $\Lambda(n)$.)

(continued on next page)

We write this identity as

$$\sum_{\substack{n\in\mathbb{N}\\ n\equiv a \pmod q}} \Lambda(n)n^{-s} = -\frac{1}{\phi(q)}\frac{L'}{L}(s,\chi_0) - \frac{1}{\phi(q)}\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} \overline{\chi(a)}\frac{L'}{L}(s,\chi).$$

Notice that the sum on the right-hand side is *analytic* near $s = 1$, because $L(1,\chi) \neq 0$ for all characters; in particular, that sum is $O_q(1)$. On the other hand, $\frac{L'}{L}(s,\chi_0)$ has a simple pole with residue -1 at $s = 1$, and therefore $\frac{L'}{L}(s,\chi_0) = \frac{1}{\phi(q)}\frac{1}{s-1} + O_q(1)$ near $s = 1$.

Taking $s = \sigma > 1$ and summarizing, we have shown that

$$\sum_{\substack{n\in\mathbb{N}\\ n\equiv a \pmod q}} \Lambda(n)n^{-\sigma} = \frac{1}{\phi(q)}\frac{1}{\sigma-1} + O_q(1).$$

Taking the limit of both sides as $\sigma \to 1+$, and using the monotone convergence theorem on the left-hand side, we deduce that

$$\sum_{\substack{n\in\mathbb{N}\\ n\equiv a \pmod q}} \Lambda(n)n^{-1} = \infty.$$

And one can show (see the book) that the contribution to the left-hand side from proper prime powers is bounded, and therefore

$$\sum_{\substack{p\in\mathbb{N}\\ p\equiv a \pmod q}} \frac{\log p}{p} = \infty. \tag{3}$$

In particular, *there must be infinitely many primes p congruent to a* (mod q), for any integers q and a with $(a,q) = 1$. This is Dirichlet's theorem on primes in arithmetic progressions (Corollary 4.10), proved in the 1830s.

Exercise: let $\pi(x;q,a)$ denote the number of primes $p \le x$ such that $p \equiv a$ (mod q)*. Prove that $\pi(x;q,a) = \Omega(x/(\log x)^{2+\varepsilon})$ for every fixed $\varepsilon > 0$. Hint: assume the contrary and use partial summation to contradict equation* (3)*.*

Remark: we saw in this proof how important it was that $L(1,\chi) \neq 0$ for all characters χ. I believe that this nonvanishing assertion is actually equivalent to the statement that there are infinitely many primes in every reduced residue class.

For the last bit of reading, go through pages 122–123 to see this argument laid out with a few more details.