**Weighted sums over primes in arithmetic progressions.** On Tuesday we proved Dirichlet's theorem on primes in arithmetic progressions: if $q \in \mathbb{N}$ and $a \in \mathbb{Z}$ satisfy $(a,q)=1$, then

$$\lim_{x\to\infty} \sum_{\substack{p\le x \\ p\equiv a \ (\mathrm{mod}\ q)}} \frac{\log p}{p} = \infty,$$

and thus in particular there are infinitely many primes in the residue class $a \pmod q$. (You should be able to quickly convince yourself that the hypothesis $(a,q)=1$ is necessary for this conclusion to hold.) In fact we can actually derive an asymptotic formula for the sum on the left-hand side. Indeed, we can get analogues of all of Mertens's asymptotic formulas restricted to one of these residue classes.

The key starting point is to show that for any nonprincipal character $\chi$,

$$L'(1,\chi) = -\sum_{n=1}^{\infty} \frac{\chi(n)\log n}{n} = -\sum_{n\le x} \frac{\chi(n)\log n}{n} + O_\chi\biggl(\frac{\log x}{x}\biggr).$$

The first equality follows from the fact that $L'(s,\chi)$ is represented by the term-by-term derivative of the Dirichlet series for $L(1,\chi)$ when $\sigma > 0$, and the second estimate follows from one of our standard partial-summation arguments.

From here, we can use the convolution identity $\log = \Lambda * 1$ and proceed just as we did for the original Mertens's formulas, finding estimates for

$$\sum_{n\le x} \frac{\chi(n)\Lambda(n)}{n}, \quad \sum_{p\le x} \frac{\chi(p)\log p}{p}, \quad \sum_{p\le x} \frac{\chi(p)}{p}, \quad \text{and} \prod_{p\le x}\biggl(1-\frac{\chi(p)}{p}\biggr)^{-1}.$$

(All the resulting estimates differ from the classical Mertens formulas in that there are no "main term" functions of $x$ that tend to infinity.) From these formulas, we can use orthogonality (that is, using a linear combination of Dirichlet characters $\pmod q$ to detect a reduced residue class $a \pmod q$) to deduce asymptotic formulas for the related expressions

$$\sum_{\substack{n\le x \\ n\equiv a \ (\mathrm{mod}\ q)}} \frac{\Lambda(n)}{n}, \quad \sum_{\substack{p\le x \\ p\equiv a \ (\mathrm{mod}\ q)}} \frac{\log p}{p}, \quad \sum_{\substack{p\le x \\ p\equiv a \ (\mathrm{mod}\ q)}} \frac{1}{p}, \quad \text{and} \prod_{\substack{p\le x \\ p\equiv a \ (\mathrm{mod}\ q)}} \biggl(1-\frac{1}{p}\biggr)^{-1}.$$

In particular, we find that uniformly over reduced residue classes $a \pmod q$,

$$\sum_{\substack{p\le x \\ p\equiv a \ (\mathrm{mod}\ q)}} \frac{\log p}{p} = \frac{1}{\phi(q)}\log x + O_q(1).$$

In light of Mertens's formula $\sum_{p\le x} \frac{\log p}{p} = \log x + O(1)$, this can be interpreted as saying that, at least when weighted by $\frac{\log p}{p}$, the primes are equally distributed among the $\phi(q)$ reduced residue classes $\pmod q$.

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Remark: If $\pi(x; q, a)$ denotes the number of primes up to $x$ that are congruent to $a$ (mod $q$), it does turn out to be true that $\pi(x; q, a) \sim \frac{1}{\phi(q)} \operatorname{li}(x)$, and so the primes really are asymptotically equally distributed among the reduced residue classes. (For example, the special case $q = 10$ tells us that asymptotically 25% of the primes have $1$ as their last digit, and similarly for last digits $3$, $7$, and $9$.) This is the *prime number theorem for arithmetic progressions*, which will be the first major result in the topics course I plan on teaching in the fall. However, just like the prime number theorem does not follow from Mertens's formulas (you checked this yourself earlier in the semester), this prime number theorem in arithmetic progressions is much deeper than the results of today's lecture.

*Not much reading for today—just Theorem 4.11 and Corollary 4.12 and their proofs.*