Math 539—Group Work #3
Thursday, January 30, 2020

Group work criteria: Start from the top and understand one problem fully before moving on to the next one; quality is more important than quantity (although these group work problems are designed so that ideally you will be able to finish them all). I will be going from group to group during the hour, paying attention to the following aspects.

1. Effective communication—including both listening and speaking, with respect for other people and their ideas
2. Engagement with, and curiosity about, the material (for instance, how far might something generalize?)
3. Boldness—suggesting ideas, and trying plans even when they're incomplete
4. Obtaining valid solutions (which are understood by everyone in the group) to the given problems

1. Recall our standard prime-counting functions (where p always denotes a prime):

$$\pi(x) = \#\{p \le x\} = \sum_{p \le x} 1, \qquad \theta(x) = \sum_{p \le x} \log p, \qquad \psi(x) = \sum_{n \le x} \Lambda(n) = \sum_{p^r \le x} \log p$$

(a) *Without using Dirichlet series, prove that* $\Lambda = \mu * \log$.
(b) *Prove that* $\operatorname{lcm}[1, 2, \dots, n] = e^{\psi(n)}$ *(exactly!).*

(a) By Möbius inversion, it suffices to prove that $\log = \Lambda * 1$. But if $n = p_1^{r_1} \cdots p_k^{r_k}$ is the factorization of n into the product of powers of distinct primes (so that $k = \omega(n)$), then

$$(\Lambda * 1)(n) = \sum_{d \mid n} \Lambda(d) = \sum_{p^r \mid n} \log p = \sum_{i=1}^{k} \sum_{j=1}^{r_i} \log p_i = \sum_{i=1}^{k} r_i \log p_i = \log \prod_{i=1}^{k} p_i^{r_i} = \log n.$$

(b) It's equivalent to prove that $\psi(n) = \log \operatorname{lcm}[1, 2, \dots, n]$. For each prime p, the power of p dividing $\operatorname{lcm}[1, 2, \dots, n]$ is exactly the largest power of p not exceeding n, which is $p^{\lfloor (\log n)/\log p \rfloor}$. Therefore

$$\begin{aligned}
\log \operatorname{lcm}[1, 2, \dots, n] = \log \prod_p p^{\lfloor (\log n)/\log p \rfloor} &= \sum_p \left\lfloor \frac{\log n}{\log p} \right\rfloor \log p \\
&= \sum_p \log p \sum_{r \le (\log n)/\log p} 1 = \sum_{p^r \le n} \log p = \psi(n).
\end{aligned}$$

(One can also argue that $\psi(x)$ and $\log \operatorname{lcm}[\{n \le x\}]$ are piecewise constant functions, both with value 0 at $x = 1$, and both with jump discontinuities exactly at $x = p^r$, for prime powers p^r, of size $\log p$.)

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2.

(a) *Prove that* $\theta(x) = \psi(x) + O(\sqrt{x})$*, and deduce that* $\theta(x) \asymp x$*.*

(b) *Prove that* $\pi(x) = \dfrac{\theta(x)}{\log x} + O\left(\dfrac{x}{\log^2 x}\right)$*, and deduce that* $\pi(x) \asymp \dfrac{x}{\log x}$*.*

(c) *Conclude that* $\psi(x) = \pi(x)\log x + O\left(\dfrac{x}{\log x}\right)$*, and that the three statements*

$$\pi(x) \sim \frac{x}{\log x}, \qquad \theta(x) \sim x, \qquad \psi(x) \sim x$$

are all equivalent.

(a) Note the convenient identity

$$\psi(x) = \sum_{p^r \le x} \log p = \sum_r \sum_{p \le x^{1/r}} \log p = \sum_r \theta(x^{1/r}) = \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + \cdots.$$

Since $\theta(y) = 0$ for all $y < 2$, the rth summand vanishes for $r > (\log x)/\log 2$, and thus

$$\psi(x) = \theta(x) + O\left(\theta(x^{1/2}) + \sum_{r=3}^{\lfloor(\log x)/\log 2\rfloor} \theta(x^{1/r})\right).$$

Trivially $\theta(y) \le \psi(y)$, and Chebyshev's theorem gives in particular $\psi(y) \ll y$; therefore

$$\psi(x) = \theta(x) + O\left(x^{1/2} + \sum_{r=3}^{\lfloor(\log x)/\log 2\rfloor} x^{1/r}\right) = \theta(x) + O(x^{1/2} + x^{1/3}\log x) = \theta(x) + O(x^{1/2}).$$

Finally, Chebyshev's theorem $\psi(x) \asymp x$ implies that $\theta(x) \asymp x + O(\sqrt{x})$, which is the same as $\theta(x) \asymp x$.

(b) Using partial summation,

$$\begin{aligned}\pi(x) &= \sum_{p \le x} \log p \cdot \frac{1}{\log p} \\ &= \int_{2-}^{x} \frac{1}{\log u}\, d\theta(u) = \frac{\theta(u)}{\log u}\bigg|_{2-}^{x} - \int_{2-}^{x} \theta(u)\, d\frac{1}{\log u} = \frac{\theta(x)}{\log x} - 0 + \int_2^x \frac{\theta(u)}{u \log^2 u}\, du.\end{aligned}$$

Since $\theta(u) \ll u$ by Chebyshev's theorem as above, we may thus write

$$\pi(x) = \frac{\theta(x)}{\log x} + O\left(\int_2^x \frac{1}{\log^2 u}\, du\right).$$

The easiest possible bound for this integral—maximum value of the integrand times length of the interval of integration—only gives $O(x)$, which is true but unhelpful. However, cutting the integral into two pieces and using this easy bound on both pieces works: for any $2 \le y \le x$,

$$\int_2^x \frac{1}{\log^2 u}\, du = \int_2^y \frac{1}{\log^2 u}\, du + \int_y^x \frac{1}{\log^2 u}\, du \le (y-2)\frac{1}{\log^2 2} + (x-y)\frac{1}{\log^2 y} \ll y + \frac{x}{\log^2 y},$$

and lots of choices of y (for example, $y = \sqrt{x}$ or $y = x/\log^2 x$) result in the estimate $\displaystyle\int_2^x \frac{1}{\log^2 u}\, du \ll \frac{x}{\log^2 x}$ that we need to finish the proof. Finally, the result $\theta(x) \asymp x$ from part (a) yields

$$\pi(x) \asymp \frac{x}{\log x} + O\biggl(\frac{x}{\log^2 x}\biggr), \quad \text{or equivalently } \pi(x) \asymp \frac{x}{\log x}.$$

(c) Solving for $\theta(x)$ in the formula from part (b) gives

$$\theta(x) = \pi(x)\log x + O\biggl(\frac{x}{\log x}\biggr),$$

and therefore part (a) gives

$$\psi(x) = \theta(x) + O(\sqrt{x}) = \pi(x)\log x + O\biggl(\frac{x}{\log x} + \sqrt{x}\biggr) = \pi(x)\log x + O\biggl(\frac{x}{\log x}\biggr).$$

The equivalence of the three given asymptotic formulas is now easy from all the relations we have. For example, if we assume that $\theta(x) \sim x$, then using part (b) gives

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = \lim_{x\to\infty} \frac{\theta(x)/\log x + O(x/\log^2 x)}{x/\log x} = \lim_{x\to\infty} \biggl(\frac{\theta(x)}{x} + O\biggl(\frac{1}{\log x}\biggr)\biggr) = 1 + 0,$$

which means that $\pi(x) \sim x/\log x$.

3. *In this problem, "the prime number theorem" refers to the statement that $\pi(x) \sim x/\log x$.*

(a) *Assuming the prime number theorem, prove that $\sum_{p\le x} \frac{1}{p} \sim \log\log x$.*

(b) *Mertens's formula (Montgomery & Vaughan, Theorem 2.7(d)) states that*

$$\sum_{p\le x} \frac{1}{p} = \log\log x + b + O\biggl(\frac{1}{\log x}\biggr), \tag{1}$$

where b is a particular constant. Starting from this formula, use partial summation to see what can be deduced about $\pi(x)$. Can you prove in this way that $\pi(x) \ll x/\log x$? that $\pi(x) \gg x/\log x$? Can you prove the prime number theorem?

(a) By partial summation,

$$\sum_{p\le x} \frac{1}{p} = \int_{2-}^{x} \frac{1}{u}\,d\pi(u) = \frac{\pi(u)}{u}\bigg|_{2-}^{x} - \int_{2-}^{x} \pi(u)\,d\frac{1}{u} = \frac{\pi(x)}{x} - 0 + \int_{2-}^{x} \frac{\pi(u)}{u^2}\,du.$$

Given $\varepsilon > 0$, choose a real number $c = c(\varepsilon)$ such that $\pi(x) < (1+\varepsilon)x/\log x$ for all $x > c$; then when $x > c$,

$$\begin{aligned}\sum_{p\le x} \frac{1}{p} &< \frac{(1+\varepsilon)x/\log x}{x} + \int_{2}^{c} \frac{\pi(u)}{u^2}\,du + \int_{c}^{x} \frac{(1+\varepsilon)u/\log u}{u^2}\,du \\ &= O_\varepsilon\biggl(\frac{1}{\log x}\biggr) + O_\varepsilon(1) + (1+\varepsilon)\int_{c}^{x} \frac{du}{u\log u} = (1+\varepsilon)\log\log x + O_\varepsilon(1).\end{aligned}$$

By a similar argument using $\pi(x) > (1-\varepsilon)x/\log x$ for sufficiently large x, we find that

$$\sum_{p\le x} \frac{1}{p} > (1-\varepsilon)\log\log x + O_\varepsilon(1).$$

Since the above inequalities are true for all $\varepsilon > 0$, they are enough to show that $\sum_{p\le x} \frac{1}{p} \sim \log\log x$.

(b) Define $M(x) = \sum_{p \le x} \frac{1}{p}$ and $R(x) = M(x) - (\log\log x + b)$, so that $R(x) \ll 1/\log x$ by assumption. Using partial summation, we try expressing $\pi(x)$ as

$$\begin{aligned}
\pi(x) = \sum_{p \le x} \frac{1}{p} \cdot p = \int_{2-}^{x} u\, dM(u) &= \int_{2-}^{x} u\, d\big(\log\log u + b + R(u)\big) \\
&= \int_{2-}^{x} u\, d\big(\log\log u + b\big) + \int_{2-}^{x} u\, dR(u) \\
&= \int_{2}^{x} \frac{du}{\log u} + xR(x) - \int_{2}^{x} R(u)\, du.
\end{aligned}$$

Using $R(x) \ll 1/\log x$, we obtain

$$\pi(x) = \int_2^x \frac{du}{\log u} + O\bigg(\frac{x}{\log x} + \int_2^x \frac{du}{\log u} \bigg) \ll \frac{x}{\log x} + \int_2^x \frac{du}{\log u}$$

(the "main term" has disappeared into the error term). The trick used in #2(b) above (splitting into two pieces and using trivial bounds on each piece) shows that $\int_2^x du/\log u \ll x/\log x$, and so we have deduced that $\pi(x) \ll x/\log x$.

However, if $R(x)$ were about $-100/\log x$, say, then the above argument gives a *negative* lower bound (since we can't predict the sign of $\int_2^x R(u)\, du$ from the sign of $R(x)$ alone); hence we cannot prove $\pi(x) \gg x/\log x$, much less the prime number theorem from the given information on $M(x)$.

Remark: we will see later in the semester (and you could actually derive for yourself) that $\int_2^x du/\log u \sim x/\log x$. If we actually assumed the stronger statement

$$\sum_{p \le x} \frac{1}{p} = \log\log x + b + o\bigg(\frac{1}{\log x}\bigg), \tag{2}$$

then the above partial summation argument actually would show that

$$\pi(x) = \int_2^x \frac{du}{\log u} + o\bigg(\frac{x}{\log x}\bigg) \sim \frac{x}{\log x}.$$

Indeed, it is known that the seemingly tiny strengthening (2) of Mertens's theorem (1) is actually *equivalent* to the prime number theorem.