Math 539—Group Work #5

Thursday, February 13, 2020

In all these problems, ε, σ_0, T, U, x, and y are positive real numbers with $\varepsilon < T$ and $U > \sigma_0$, and $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is a Dirichlet series with finite abscissa of convergence.

1. Suppose $y \ge 2$. By considering the contour integral $\frac{1}{2\pi i} \oint_{R-} \frac{y^s}{s}\, ds$, where $R-$ is the rectangle with corners at $\sigma_0 - iT$, $\sigma_0 + iT$, $-U + iT$, and $-U - iT$ (oriented counterclockwise), show that

$$1 = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{y^s}{s}\, ds + O\biggl(\frac{y^{\sigma_0}}{T \log y} + \frac{T}{U} y^{-U} \biggr). \tag{1}$$

Conclude that uniformly for $y \ge 2$,

$$\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{y^s}{s}\, ds = 1 + O\biggl(\frac{y^{\sigma_0}}{T} \biggr).$$

The only pole of the integrand is at $s = 0$, at which the residue is $y^0 = 1$; since this pole lies inside the closed path of integration, the contour integral equals 1 by the residue theorem; therefore

$$1 = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{y^s}{s}\, ds + O\biggl(\biggl| \frac{1}{2\pi i} \int_{\sigma_0 + iT}^{-U + iT} \frac{y^s}{s}\, ds \biggr| + \biggl| \frac{1}{2\pi i} \int_{-U + iT}^{-U - iT} \frac{y^s}{s}\, ds \biggr| + \biggl| \frac{1}{2\pi i} \int_{-U - iT}^{\sigma_0 - iT} \frac{y^s}{s}\, ds \biggr| \biggr).$$

On the other hand, the contribution to the contour integral from the top edge of $R-$ is at most

$$\biggl| -\frac{1}{2\pi i} \int_{-U + iT}^{\sigma_0 + iT} \frac{y^s}{s}\, ds \biggr| \le \int_{-U}^{\sigma_0} \frac{y^\sigma}{T}\, d\sigma < \int_{-\infty}^{\sigma_0} \frac{y^\sigma}{T}\, d\sigma = \frac{y^{\sigma_0}}{T \log y},$$

where the lower boundary term vanishes since $y > 1$. The same bound holds for the contribution from the bottom edge of $R-$. Finally, the contribution from the left edge of $R-$ is at most

$$\biggl| -\frac{1}{2\pi i} \int_{-U - iT}^{-U + iT} \frac{y^s}{s}\, ds \biggr| \le \int_{-T}^{T} \frac{y^{-U}}{U}\, dt = \frac{2T}{U} y^{-U},$$

which completes the proof of equation (1). Since that equation is uniform in U, we may take the limit as $U \to \infty$ to obtain

$$\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{y^s}{s}\, ds = 1 + O\biggl(\frac{y^{\sigma_0}}{T \log y} \biggr) = 1 + O\biggl(\frac{y^{\sigma_0}}{T} \biggr),$$

where we have used $y \ge 2$ in the last step.

(Some people would phrase this argument directly as an integral over the infinite contour from $-\infty - iT$ to $\sigma_0 - iT$ to $\sigma_0 + iT$ to $-\infty + iT$. However, the legitimacy of the residue theorem over noncompact regions such as this half-infinite strip requires some decay condition on the integrand, which is basically equivalent to the argument above.)

(continued on next page)

2. Suppose $0 < y \le \frac{1}{2}$. *By considering the contour integral* $\frac{1}{2\pi i}\oint_{R+} \frac{y^s}{s}\,ds$, *where* $R+$ *is the rectangle with corners at* $\sigma_0 - iT$, $\sigma_0 + iT$, $U + iT$, *and* $U - iT$, *show that*

$$0 = \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{y^s}{s}\,ds + O\biggl(\frac{y^{\sigma_0}}{T|\log y|} + \frac{T}{U}y^U\biggr). \tag{2}$$

Conclude that uniformly for $0 < y \le \frac{1}{2}$,

$$\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{y^s}{s}\,ds = O\biggl(\frac{y^{\sigma_0}}{T}\biggr).$$

The only pole of the integrand is again at $s = 0$, but now that pole is outside the closed path of integration; therefore the contour integral equals 0 by the residue theorem, and so

$$0 = \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{y^s}{s}\,ds + O\biggl(\biggl|\frac{1}{2\pi i}\int_{\sigma_0 + iT}^{U + iT} \frac{y^s}{s}\,ds\biggr| + \biggl|\frac{1}{2\pi i}\int_{U + iT}^{U - iT} \frac{y^s}{s}\,ds\biggr| + \biggl|\frac{1}{2\pi i}\int_{U - iT}^{\sigma_0 - iT} \frac{y^s}{s}\,ds\biggr|\biggr).$$

(As written, we are traversing the contour clockwise, but that's insignificant since $-0 = 0$.) The contribution from the top edge of $R+$ is at most

$$\biggl|\frac{1}{2\pi i}\int_{\sigma_0 + iT}^{U + iT} \frac{y^s}{s}\,ds\biggr| \le \int_{\sigma_0}^{U} \frac{y^\sigma}{T}\,d\sigma < \int_{\sigma_0}^{\infty} \frac{y^\sigma}{T}\,d\sigma = -\frac{y^{\sigma_0}}{T\log y} = \frac{y^{\sigma_0}}{T|\log y|},$$

where the lower boundary term vanishes since $0 < y < 1$. The same bound holds for the contribution from the bottom edge of $R+$. Finally, the contribution from the right edge of $R+$ is at most

$$\biggl|\frac{1}{2\pi i}\int_{U - iT}^{\sigma_0 - iT} \frac{y^s}{s}\,ds\biggr| \le \int_{-T}^{T} \frac{y^U}{U}\,dt = \frac{2T}{U}y^U,$$

which completes the proof of equation (2). Since that equation is uniform in U, we may take the limit as $U \to \infty$ to obtain

$$\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{y^s}{s}\,ds = 1 + O\biggl(\frac{y^{\sigma_0}}{T|\log y|}\biggr) = 1 + O\biggl(\frac{y^{\sigma_0}}{T}\biggr),$$

where we have used $0 < y \le \frac{1}{2}$ in the last step.

Define the "sine integral" $\operatorname{si}(x) = -\int_x^\infty \frac{\sin u}{u}\,du$. *You may use (without proof) the facts that* $\operatorname{si}(x)$ *is bounded and* $\operatorname{si}(x) + \operatorname{si}(-x) = \lim_{x\to-\infty}\operatorname{si}(x) = -\pi$.

We note for later use that $\operatorname{si}(0) = -\pi/2$, since $\operatorname{si}(0) + \operatorname{si}(-0) = -\pi$.

3. Show that for all $\sigma_0 > \max\{0, \sigma_a\}$,

$$\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s)\frac{x^s}{s}\,ds = \sideset{}{'}\sum_{n\le x} a_n + \sum_{x/2<n<x} \frac{a_n}{\pi}\operatorname{si}\left(T\log\frac{x}{n}\right) - \sum_{x<n<2x} \frac{a_n}{\pi}\operatorname{si}\left(T\log\frac{n}{x}\right) + O\biggl(\frac{(4x)^{\sigma_0}}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma_0}}\biggr).$$

You may use the result in equation (5) *below to solve #3.*

[Remark: this statement is Theorem 5.2 in Montgomery and Vaughan's book.] Since the Dirichlet series $\alpha(s)$ converges absolutely on the path of integration by assumption, we may exchange the integral and the infinite sum (by Fubini's theorem), obtaining

$$\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\alpha(s)\frac{x^s}{s}\,ds=\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\sum_{n=1}^{\infty}a_n n^{-s}\frac{x^s}{s}\,ds=\sum_{n=1}^{\infty}a_n\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\frac{(x/n)^s}{s}\,ds.$$

We split this sum into four subsums, writing the right-hand side as

$$\sum_{n\le x/2}\frac{a_n}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\frac{(x/n)^s}{s}\,ds+\sum_{x/2<n\le x}\frac{a_n}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\frac{(x/n)^s}{s}\,ds$$
$$+\sum_{x<n<2x}\frac{a_n}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\frac{(x/n)^s}{s}\,ds+\sum_{n\ge 2x}\frac{a_n}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\frac{(x/n)^s}{s}\,ds. \quad (3)$$

In the first subsum, $x/n\ge 2$, and so problem #1 gives

$$\sum_{n\le x/2}\frac{a_n}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\frac{(x/n)^s}{s}\,ds=\sum_{n\le x/2}a_n\left(1+O\left(\frac{(x/n)^{\sigma_0}}{T}\right)\right);$$

similarly, in the last subsum, $0<x/n\le 1/2$, and so problem #2 gives

$$\sum_{n\ge 2x}\frac{a_n}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\frac{(x/n)^s}{s}\,ds=\sum_{n\le x/2}a_n O\left(\frac{(x/n)^{\sigma_0}}{T}\right).$$

For the second subsum in equation (3), $1\le x/n<2$, and so equation (5) below gives

$$\begin{aligned}\sum_{x/2<n\le x}\frac{a_n}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\frac{(x/n)^s}{s}\,ds&=\sum_{x/2<n\le x}a_n\left(1+\frac{1}{\pi}\operatorname{si}\left(T\log\frac{x}{n}\right)+O\left(\frac{2^{\sigma_0}}{T}\right)\right)\\&=\sum_{x/2<n\le x}a_n\left(1+\frac{1}{\pi}\operatorname{si}\left(T\log\frac{x}{n}\right)+O\left(\frac{2^{\sigma_0}(x/n)^{\sigma_0}}{T}\right)\right)\end{aligned}$$

(since $x/n\ge 1$); similarly, for the third subsum we have

$$\begin{aligned}\sum_{x<n<2x}\frac{a_n}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\frac{(x/n)^s}{s}\,ds&=\sum_{x<n<2x}a_n\left(-\frac{1}{\pi}\operatorname{si}\left(T\log\frac{n}{x}\right)+O\left(\frac{2^{\sigma_0}}{T}\right)\right)\\&=\sum_{x/2<n\le x}a_n\left(1+\frac{1}{\pi}\operatorname{si}\left(T\log\frac{x}{n}\right)+O\left(\frac{2^{\sigma_0}(2x/n)^{\sigma_0}}{T}\right)\right)\end{aligned}$$

(since $2x/n>1$). Combining all of these calculations is enough to complete the problem, once we note that if x is an integer then the summand $n=x$ becomes $a_n(1+\operatorname{si}(0)/\pi)=a_n/2$, as needed for the $\sum'$ notation.

4. Let S be the closed contour (oriented counterclockwise) consisting of the line segments joining the points $-i\varepsilon$, $-iT$, σ_0-iT, σ_0+iT, iT, and $i\varepsilon$ together with the right half-circle with diameter running from $i\varepsilon$ to $-i\varepsilon$.

Suppose $\frac{1}{2} < y < 2$. By considering the contour integral $\frac{1}{2\pi i}\oint_S \frac{y^s}{s}\,ds$ and letting $\varepsilon \to 0+$, show that

$$0 = \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{y^s}{s}\,ds - \left(\frac{1}{2} + \frac{1}{\pi}\operatorname{si}(T\log y)\right) - \frac{1}{2} + O\left(\frac{2^{\sigma_0}}{T}\right). \tag{4}$$

Conclude that uniformly for $\frac{1}{2} < y < 2$,

$$\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{y^s}{s}\,ds = \begin{cases} 1 + \frac{1}{\pi}\operatorname{si}(T\log y) + O\big(2^{\sigma_0}/T\big), & \text{if } 1 \le y \le 2, \\ -\frac{1}{\pi}\operatorname{si}(T\log y^{-1}) + O\big(2^{\sigma_0}/T\big), & \text{if } \frac{1}{2} \le y \le 1. \end{cases} \tag{5}$$

As above, there are no poles of the integrand inside the closed contour of integration, and therefore the contour integral equals 0. The integral in equation (4) is the contribution of the right edge of S to the contour integral. The contribution of the semicircle, using the parametrization $z = \varepsilon e^{i\theta}$, is

$$\frac{1}{2\pi i}\int_{\pi/2}^{-\pi/2} \frac{y^{\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}}\,d(\varepsilon e^{i\theta}) = -\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} y^{\varepsilon e^{i\theta}}\,d\theta;$$

as ε tends to 0, the integrand tends to 1 uniformly in θ, and therefore this contribution tends to $(-1/2\pi)(\pi/2 - (-\pi/2)) = -1/2$. (This little technical trick is the way to deal with a contour integral that wants to run through a pole of the integrand; heuristically, we see that such a pole is counted as "half inside" the contour in terms of its contribution to the residue theorem.)

The contribution from the left edge of S, using the parametrization $z = it$ (and then, in the second integral, the change of variables $t \mapsto -t$), is

$$\begin{aligned} \frac{1}{2\pi i}\int_T^{\varepsilon} \frac{y^{it}}{it}\,d(it) + \frac{1}{2\pi i}\int_{-\varepsilon}^{-T} \frac{y^{it}}{it}\,d(it) &= -\frac{1}{2\pi i}\int_{\varepsilon}^{T} \frac{y^{it}}{t}\,dt + \frac{1}{2\pi i}\int_{\varepsilon}^{T} \frac{y^{-it}}{t}\,dt \\ &= -\frac{1}{\pi}\int_{\varepsilon}^{T} \frac{e^{it\log y} - e^{-it\log y}}{2it}\,dt \\ &= -\frac{1}{\pi}\int_{\varepsilon}^{T} \frac{\sin(it\log y)}{t}\,dt = \frac{1}{\pi}\big(\operatorname{si}(\varepsilon\log y) - \operatorname{si}(T\log y)\big), \end{aligned}$$

which tends to $-1/2 - \operatorname{si}(T\log y)/\pi$ as ε tends to 0, since $\operatorname{si}(0) = -\pi/2$.

Finally, since $y < 2$, the contribution of the bottom edge of S to the contour integral is at most

$$\left|\frac{1}{2\pi i}\int_{-iT}^{\sigma_0 - iT} \frac{y^s}{s}\,ds\right| \le \int_0^{\sigma_0} \frac{y^{\sigma}}{T}\,d\sigma < \int_0^{\sigma_0} \frac{2^{\sigma}}{T}\,d\sigma < \int_{-\infty}^{\sigma_0} \frac{2^{\sigma}}{T}\,d\sigma = \frac{2^{\sigma_0}}{T\log 2},$$

and similarly for the contribution of the top edge of S to the contour integral. (This method was motivated by the desire to avoid dividing by $\log y$ when y is close to 1.) Together, these calculations establish equation (4). Deriving equation (5) from equation (4) is just a matter of rearranging terms and noting that

$$1 + \frac{1}{\pi}\operatorname{si}(T\log y) = -\frac{1}{\pi}\big(-\pi - \operatorname{si}(T\log y)\big) = -\frac{1}{\pi}\operatorname{si}(-T\log y) = -\frac{1}{\pi}\operatorname{si}(T\log y^{-1}).$$