Math 539—"Group" Work #8

Tuesday, March 17, 2020

For all of these questions, we define the Bernoulli polynomials $B_k(x)$ *as coefficients in the power series expansion*

$$f(z,x) = \frac{ze^{xz}}{e^z-1} = \sum_{k=0}^{\infty} B_k(x)\frac{z^k}{k!} = 1 + \left(x - \frac{1}{2}\right)z + \left(x^2 - x + \frac{1}{6}\right)\frac{z^2}{2} + \cdots . \tag{1}$$

We also define the Bernoulli numbers $B_k = B_k(0)$*, a few of which have been listed below.*

k	1	2	3	4	5	6	7	8	9	10	11	12
B_k	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$

Finally, recall our notation for the fractional part $\{x\} = x - \lfloor x \rfloor$.

On this group work, you may differentiate or integrate infinite series term by term with impunity (that is, don't worry about convergence issues on this group work).

1. Preliminaries:

(a) *Show that* $\int_0^1 f(z,x)\,dx = 1$. *Conclude that* $\int_0^1 B_k(x)\,dx = 0$ *for all* $k \ge 1$.

(b) *Verify the identity* $\frac{\partial f(z,x)}{\partial x} = zf(z,x)$*, and conclude that* $B_k'(x) = kB_{k-1}(x)$ *for all* $k \ge 1$.

(c) *Show that* $f(z,0) + z/2$ *is an even function of* z. *Conclude that* $B_1 = -1/2$ *and that* $B_{2j+1} = 0$ *for all* $j \ge 1$.

(d) *Prove that* $B_k(1) = B_k$ *for all* $k \ge 2$*, and conclude that* $B_k\big(\{x\}\big)$ *is a continuous periodic function with period* 1 *for all* $k \ge 2$. *(Hint: part (b) has something to say about the difference* $B_k(1) - B_k(0)$*.)*

(e) *Why do parts (a), (b), and (d) imply that* $B_{k+1}(\{x\})/(k+1)$ *is an antiderivative for* $B_k\big(\{x\}\big)$ *on the entire real line, for every* $k \ge 1$*?*

(a) Integrating both sides of equation (1) with respect to x yields

$$\int_0^1 f(z,x)\,dx = \int_0^1 \frac{ze^{xz}}{e^z-1}\,dx = \frac{z}{e^z-1}\frac{e^{xz}}{z}\bigg|_0^1 = \sum_{k=0}^{\infty}\frac{z^k}{k!}\int_0^1 B_k(x)\,dx = \frac{z}{e^z-1}\frac{e^z-1}{z} = 1$$

for $z \ne 0$, and the claim is trivial for $z = 0$ since $f(0,x) = 1$ identically. Furthermore,

$$1 = \int_0^1 f(z,x)\,dx = \int_0^1 \sum_{k=0}^{\infty} B_k(x)\frac{z^k}{k!}\,dx = \sum_{k=0}^{\infty}\frac{z^k}{k!}\int_0^1 B_k(x)\,dx = \sum_{k=0}^{\infty} z^k\left(\frac{1}{k!}\int_0^1 B_k(x)\,dx\right).$$

By uniqueness of power series expansions, the coefficient $\frac{1}{k!}\int_0^1 B_k(x)\,dx$ of x^k on the right-hand side equals the coefficient 0 of z^k on the left-hand side for all $k \ge 1$, which gives the desired evaluation. (Integrating term by term is valid because, as one can check, the series converges uniformly for $0 \le x \le 1$ for any fixed z inside the disc of convergence $\{|z| < 2\pi\}$ of the series.)

(b) The verification is simple:

$$\frac{\partial f(z,x)}{\partial x} = \frac{z}{e^z-1}\frac{\partial e^{xz}}{\partial x} = \frac{z}{e^z-1} z e^{zx} = z f(z,x).$$

Writing this identity in terms of the series gives

$$\frac{\partial}{\partial x}\sum_{k=0}^{\infty} B_k(x)\frac{z^k}{k!} = z\sum_{k=0}^{\infty} B_k(x)\frac{z^k}{k!}$$
$$\sum_{k=0}^{\infty} B_k'(x)\frac{z^k}{k!} = \sum_{k=0}^{\infty} B_k(x)\frac{z^{k+1}}{k!} = \sum_{k=1}^{\infty} B_{k-1}(x)\frac{z^k}{(k-1)!}.$$

Again by uniqueness of power series expansions, we conclude that $\frac{B_k'(x)}{k!} = \frac{B_{k-1}(x)}{(k-1)!}$ for all $k \ge 1$, which gives the desired identity.

(c) We have

$$f(z,0) + \frac{z}{2} = \frac{z}{e^z-1} + \frac{z}{2} = \frac{2z + z(e^z-1)}{2(e^z-1)} = \frac{z(e^z+1)}{e^z-1} = \frac{z(e^{z/2}+e^{-z/2})}{e^{z/2}-e^{-z/2}},$$

which we check is invariant under changing z to $-z$. Therefore all of the odd power series coefficients of $f(z,0) + \frac{z}{2} = -\frac{1}{2}z + \sum_{k=0}^{\infty} B_k \frac{z^k}{k!}$ equal 0, which gives the required values.

(d) By the fundamental theorem of calculus and part (b),

$$B_k(1) - B_k(0) = \int_0^1 B_k'(x)\,dx = \int_0^1 kB_{k-1}(x)\,dx,$$

which equals 0 for $k-1 \ge 1$ by part (a). Since $\{x\}$ is a periodic function with period 1, so is $B_k(\{x\})$; and whenever x is not an integer, $B_k(\{x\})$ is a composition of two continuous functions and hence is itself continuous. On the other hand, when x is an integer, then (by continuity of B_k) we have $\lim_{w\to x-} B_k(\{x\}) = B_k(\lim_{w\to x-}\{x\}) = B_k(1-) = B_k(1)$ and $\lim_{w\to x+} B_k(\{x\}) = B_k(\lim_{w\to x+}\{x\}) = B_k(0+)$, both of which equal $B_k(\{x\}) = B_k(0)$. Therefore $B_k(\{x\})$ is continuous for all real numbers x.

(e) Certainly part (b) implies that $B_{k+1}(\{x\})/(k+1)$ is an antiderivative for $B_k(\{x\})$ for $0 < x < 1$, since $\{x\} = x$ there. On the other hand, part (a) implies that $\int_0^x B_k(\{u\})\,du = \int_{\lfloor x\rfloor}^x B_k(\{u\})\,du = \int_0^{\{x\}} B_k(u)\,du$, so that this antiderivative must be periodic with period 1. Part (d) then implies that $B_{k+1}(\{x\})/(k+1)$ is the appropriate antiderivative for all $x \in \mathbb{R}$.

Remark: Although not relevant to this group work, it is a wonderful fact that the Bernoulli polynomials also provide the formulas for the sum of the first N powers of integers, generalizing the well-known formulas for the sum of the first N integers, squares, or cubes:

$$\sum_{0\le n<N} n^k = \frac{B_{k+1}(N) - B_{k+1}}{k+1} = \int_0^N B_k(x)\,dx.$$

Thus the Bernoulli polynomial $B_{k+1}(x)$ also functions as a "discrete antiderivative" of the simple power function x^k.

For the next problem, you may use the known Fourier expansion of the sawtooth function

$$-\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin(2\pi mx)=\begin{cases}\{x\}-\frac{1}{2}, & \text{if } x \text{ is not an integer},\\ 0, & \text{if } x \text{ is an integer}.\end{cases} \tag{2}$$

2. A formula for the values of $\zeta(s)$ at positive even integers:

(a) *When $x\notin\mathbb{Z}$, show that $B_{2j}\big(\{x\}\big)=(-1)^{j-1}(2j)!\sum_{m=1}^{\infty}\frac{2\cos(2\pi mx)}{(2\pi m)^{2j}}$ for all $j\ge 1$.*

(b) *Deduce that for all $j\ge 1$,*

$$\zeta(2j)=\frac{(-1)^{j-1}2^{2j-1}\pi^{2j}B_{2j}}{(2j)!}. \tag{3}$$

Conclude that in particular, $\zeta(2j)$ equals a rational number times π^{2j} for all $j\ge 1$.

(a) We prove, by induction on $k\ge 1$, that when x is not an integer,

$$B_k\big(\{x\}\big)=(-1)^{\lfloor k/2\rfloor-1}k!\sum_{m=1}^{\infty}\begin{cases}2\sin(2\pi mx)/(2\pi m)^k, & \text{if } k \text{ is odd},\\ 2\cos(2\pi mx)/(2\pi m)^k, & \text{if } k \text{ is even};\end{cases} \tag{4}$$

the assertion follows upon taking $k=2j$. Since both expressions are periodic functions with period 1, we may assume $0<x<1$. Note that $B_1(x)=x-\frac{1}{2}$, as we see in equation (1), and therefore the case $k=1$ of equation (4) is exactly the known identity (2).

Suppose equation (4) holds for some positive integer k. Integrating both sides of the equation and using problem #1(b), we see that

$$\begin{aligned}B_{k+1}(x)&=\int(k+1)B_k(x)\,dx\\&=\int(k+1)\left((-1)^{\lfloor k/2\rfloor-1}k!\sum_{m=1}^{\infty}\begin{cases}2\sin(2\pi mx)/(2\pi m)^k, & \text{if } k \text{ is odd},\\ 2\cos(2\pi mx)/(2\pi m)^k, & \text{if } k \text{ is even}\end{cases}\right)dx\\&=C+\left((-1)^{\lfloor k/2\rfloor-1}(k+1)!\sum_{m=1}^{\infty}\begin{cases}-2\cos(2\pi mx)/(2\pi m)^{k+1}, & \text{if } k \text{ is odd},\\ 2\sin(2\pi mx)/(2\pi m)^{k+1}, & \text{if } k \text{ is even}\end{cases}\right)dx,\end{aligned}$$

which we can see is exactly the desired formula (4) in the case $k+1$, except possibly for the constant of integration C. On the other hand, integrating both sides of equation (4) from 0 to 1 results in 0 on both sides (where we have used problem #1(a) for the left-hand side), and therefore the constant of integration must be $C=0$.

Remark: The term-by-term integration can be justified by the uniform convergence of the series (4), which is easy to establish when $k\ge 2$ but not so obvious when $k=1$. However, general theorems from the subject of Fourier series exist that justify the term-by-term integration for nice enough functions, including $B_1\big(\{x\}\big)$.

(b) Taking the limit as $x\to 0+$ of both sides of the identity proved in part (a) yields

$$B_{2j}=B_{2j}(0)=(-1)^{j-1}(2j)!\sum_{m=1}^{\infty}\frac{2}{(2\pi m)^{2j}}=(-1)^{j-1}(2j)!\frac{2}{(2\pi)^{2j}}\zeta(2j)$$

for all $j\ge 1$, which is the desired identity.

If we can prove that B_k is a rational number for every $k \geq 0$, then the given formula for $\zeta(2j)$ is indeed a rational number times π^{2j}. Perhaps the easiest way to prove this is via the following logic: if $g(z,y)$ is a rational function of z and y with integer coefficients, then every derivative of g is also a rational function of z and y with integer coefficients. Therefore every derivative of $g(z,e^z)$ is a rational function of z and e^z with integer coefficients (since the chain rule only produces additional factors of e^z); in particular, every derivative of $g(z,e^z)$ evaluated at $z=0$ is a rational function of 0 and 1 with integer coefficients, that is, a rational number.

Remark: If we try to find the value of $\zeta(2j+1)$ in this way, the corresponding series in part (a) is a sine series instead of a cosine series, and when we take the limit as $x \to 0+$ we simply reprove $B_{2j+1} = 0$ for $j \geq 1$ without gaining any knowledge about $\zeta(2j+1)$.

For the final problem, we will need the notation

$$\binom{z}{k} = \frac{z(z-1)\cdots(z-(k-1))}{k!}$$

for all $z \in \mathbb{C}$ and $k \in \mathbb{Z}_{\geq 0}$. (This is exactly the usual binomial coefficient $\binom{z}{k} = \frac{z!}{k!(z-k)!}$ when $z \geq k$ is an integer; the given formula shows that we can think of $\binom{z}{k}$ as a polynomial of degree k in the complex variable z.) We also recall from equation (1.24) the formula, valid for $\sigma > 0$:

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \{x\}x^{-s-1}\,dx. \tag{5}$$

3. An alternate way to meromorphically continue $\zeta(s)$ to the entire complex plane:

(a) *Show that for all integers $K \geq 1$ and all $\sigma > 0$,*

$$\int_1^\infty B_K\big(\{x\}\big)x^{-s-K}\,dx = -\frac{B_{K+1}}{K+1} - \frac{-s-K}{K+1}\int_1^\infty B_{K+1}\big(\{x\}\big)x^{-s-(K+1)}\,dx.$$

(b) *By induction on K (or otherwise), show that for all integers $K \geq 1$ and all $\sigma > 0$,*

$$\zeta(s) = 1 + \frac{1}{s-1} - \sum_{k=1}^{K}(-1)^k\binom{-s}{k-1}\frac{B_k}{k} - (-1)^K\binom{-s}{K}\int_1^\infty B_K\big(\{x\}\big)x^{-s-K}\,dx. \tag{6}$$

(c) *Show that the integral on the right-hand side of the above equation converges for $\sigma > 1-K$. Conclude that $\zeta(s)$ can be analytically continued to the entire complex plane except for a simple pole at $s=1$.*

(d) *Prove that $\zeta(-n) \in \mathbb{Q}$ for all $n \in \mathbb{Z}_{\geq 0}$.*

(a) From problem #1(e), we know that $B_{K+1}\big(\{x\}\big)/(K+1)$ is an antiderivative for $B_K\big(\{x\}\big)$. Therefore integration by parts gives

$$\int_1^\infty B_K\big(\{x\}\big)x^{-s-K}\,dx = \frac{B_{K+1}(\{x\})}{K+1}x^{-s-K}\bigg|_1^\infty - \int_1^\infty \frac{B_{K+1}(\{x\})}{K+1}(-s-K)x^{-s-K-1}\,dx$$
$$= 0 - \frac{B_{K+1}}{K+1} - \frac{-s-K}{K+1}\int_1^\infty B_{K+1}\big(\{x\}\big)x^{-s-(K+1)}\,dx,$$

since $B_K(1) = B_K$; the upper endpoint yields 0 because the continuous, periodic function $B_K\big(\{x\}\big)$ is bounded and $\sigma > 0$ (indeed, even $\sigma > -K$ would suffice here).

(b) When we note from problem #1 that $B_1(x) = x - \frac{1}{2}$ and $B_1 = -\frac{1}{2}$, and that $\binom{-s}{1} = -s$ from its definition, we see that the base case $K = 1$ to be proved is the identity

$$\zeta(s) = 1 + \frac{1}{s-1} - \frac{1}{2} - s\int_1^\infty \left(\{x\} - \frac{1}{2}\right)x^{-s-1}\,dx,$$

which is the same as equation (5) once we observe that $s\int_1^\infty \frac{1}{2}x^{-s-1}\,dx = \frac{1}{2}$ since $\sigma > 0$.

As for the induction step, part (a) implies that

$$\begin{aligned}
-(-1)^K\binom{-s}{K}&\int_1^\infty B_K\big(\{x\}\big)x^{-s-K}\,dx \\
&= (-1)^K\binom{-s}{K}\frac{B_{K+1}}{K+1} + (-1)^K\binom{-s}{K}\frac{-s-K}{K+1}\int_1^\infty B_{K+1}\big(\{x\}\big)x^{-s-(K+1)}\,dx \\
&= -(-1)^{K+1}\binom{-s}{K}\frac{B_{K+1}}{K+1} - (-1)^{K+1}\binom{-s}{K+1}\int_1^\infty B_{K+1}\big(\{x\}\big)x^{-s-(K+1)}\,dx
\end{aligned}$$

from the definition of $\binom{-s}{K+1}$; therefore equation (6) for the case K implies equation (6) for the case $K+1$.

Remark: This identity for $\zeta(s)$ is an example of a much more general technique called Euler–Maclaurin summation (see Appendix B), in which the Bernoulli polynomials figure prominently. For example, on can get extremely good versions of Stirling's formula (approximations to $n!$) by applying Euler–Maclaurin summation to $\log(n!) = \sum_{k=1}^n \log k$.

(c) Because the continuous periodic function $B_K\big(\{x\}\big)$ is bounded (by some constant depending on K), we have the estimate

$$\int_1^\infty B_K\big(\{x\}\big)x^{-s-K}\,dx \ll_K \int_1^\infty x^{-\sigma-K}\,dx = \frac{1}{\sigma+K-1}$$

as long as $\sigma > 1-K$. Therefore equation (6) provides a meromorphic continuation of $\zeta(s)$ to the half-plane $\sigma > 1-K$, with the only singularity caused by the term $1/(s-1)$ (the binomial coefficients are simply polynomials in s). Since K can be taken as large as we wish, we obtain a (necessarily unique) analytic continuation of $\zeta(s)$ to the entire complex plane other than the pole at $s = 1$.

Remark: Parts (b) and (c) illustrate a general phenomenon about integrals whose integrand contains an oscillatory term, namely that one can often get better convergence by integrating the integral by parts—more specifically, integrating the oscillating function (here $B_K\big(\{x\}\big)$) and differentiating the other part.

(d) If we choose $K = n+1$ (or larger), then the definition of $\binom{-s}{K}$ includes a factor $-s-n$ on top, which vanishes when we set $s = -n$. Therefore

$$\zeta(-n) = 1 + \frac{1}{-n-1} - \sum_{k=1}^{n+1}(-1)^k\binom{n}{k-1}\frac{B_k}{k}.$$

We observed in problem #2(b) that the Bernoulli numbers B_k are all rational, and therefore this right-hand side is indeed a rational number.

Remark: With another half-hour or so, we could establish the exact formula

$$\zeta(-n) = (-1)^n\frac{B_{n+1}}{n+1} \tag{7}$$

with similar elementary methods that use other combinatorial properties of the Bernoulli numbers and polynomials (see Appendix B for the details). On the other hand, it is a simple exercise to verify that the formula (7) follows directly from the formula (3) if we use the functional equation for $\zeta(s)$ that we saw in class (although the case $n = 0$ of equation (7) is slightly harder than the other cases).