Thursday, April 3
Group Work #9 today

Last Thursday we proved that if $(a,q)=1$, then

$$(*) \quad \sum_{\substack{p\leq x\\ p\equiv a \pmod q}} \frac{\log p}{p} \to \infty \text{ as } x\to\infty.$$

Exercise: Let $\pi(x;q,a) = \#\{p\leq x : p\equiv a \pmod q\}$.
Prove that for any $\varepsilon>0$,

$$\pi(x;q,a) = \Omega\left(x/(\log x)^{2+\varepsilon}\right).$$

We can actually get an asymptotic formula for the left-hand side of $(*)$ and similar Mertens-like sums.

The key input to the asymptotic formulas is to show that if $\chi \pmod q$ is nonprincipal, then

$$L'(1,\chi) = -\sum_{n=1}^{\infty} \frac{\chi(n)\log n}{n}$$

$$= -\sum_{n\leq x} \frac{\chi(n)\log n}{n} + O_\chi\left(\frac{\log x}{x}\right).$$

- First equality: $L(s,\chi)$ converges for $\sigma>0$.
- Second equality: partial summation together with $S_\chi(x) = \sum_{n\leq x}\chi(n) \ll_\chi 1$.

Theorem 4.11: Let $\delta_\chi = \begin{cases} 1, & \text{if } \chi=\chi_0, \\ 0, & \text{if } \chi\neq\chi_0.\end{cases}$

Then

$$\sum_{n\leq x} \frac{\chi(n)\Lambda(n)}{n} = \delta_\chi \log x + O_\chi(1)$$

$$\sum_{p\leq x} \frac{\chi(p)\log p}{p} = \delta_\chi \log x + O_\chi(1)$$

$$\sum_{p\leq x} \frac{\chi(p)}{p} = \delta_\chi \log\log x + b(\chi) + O_\chi\left(\frac{1}{\log x}\right).$$

If $\chi\neq\chi_0$, $\prod_{p\leq x}\left(1-\frac{\chi(p)}{p}\right)^{-1} = L(1,\chi) + O_\chi\left(\frac{1}{\log x}\right).$

To go from χ- formulas to $a \pmod q$ formulas, we use the orthogonality relation

$$\begin{cases} 1, & \text{if } n \equiv a \pmod q, \\ 0, & \text{if } n \not\equiv a \pmod q \end{cases} = \frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi(a)}\, \chi(n).$$

Corollary 4.12: Assume $(a,q)=1$. Then

- $$\sum_{\substack{n \le x \\ n \equiv a \pmod q}} \frac{\Lambda(n)}{n} = \frac{1}{\phi(q)} \log x + O_q(1)$$

- $$\sum_{\substack{p \le x \\ p \equiv a \pmod q}} \frac{\log p}{p} = \frac{1}{\phi(q)} \log x + O_q(1)$$

- $$\sum_{\substack{p \le x \\ p \equiv a \pmod q}} \frac{1}{p} = \frac{1}{\phi(q)} \log\log x + b(q,a) + O_q(1/\log x)$$

- $$\prod_{\substack{p \le x \\ p \equiv a \pmod q}} \left(1 - \frac{1}{p}\right)^{-1} = c(q,a) (\log x)^{1/\phi(q)} * (1 + O_q(1/\log x)).$$

Remark: Today we know that

$$\pi(x;q,a) = \frac{1}{\phi(q)} li(x) + O(\longrightarrow).$$

But from this corollary, we can only get $\pi(x;q,a) \ll li(x)$.

$$\ll \frac{x}{\log x}.$$