Tuesday, February 11

Group Work #5 on Thursday

RECALL:

Lemma: Let $y \geq 3$ and set $n = \prod_{p \leq y} p$. Then $\omega(n) = \frac{\log n}{\log\log n}\left(1 + O\left(\frac{1}{\log\log n}\right)\right)$.

Calculus exercise: Both $\frac{\log t}{\log\log t}$ and $\frac{\log t}{(\log\log t)^2}$ are increasing for $t \geq e^{e^2} \approx 1618.2$.

Theorem 2.10: For all $m \geq 3$,

$$\omega(m) \leq \frac{\log m}{\log\log m}\left(1 + O\left(\frac{1}{\log\log m}\right)\right).$$

Proof: The inequality holds trivially* for $\omega(m) \leq 4$.

*Why? We know $\frac{\log m}{\log\log m} \geq 4$ for some $m \geq M_0$.

Consider $E(m) = \left|\omega(m) \Big/ \frac{\log m}{\log\log m} - 1\right| \cdot \log\log m$

and define $C = \max\{E(3), E(4), \ldots, E(M_0)\}$.

Thus $\left|\omega(m) \Big/ \frac{\log m}{\log\log m} - 1\right| \leq C / \log\log m$ for $m \leq M_0$.

Suppose $\omega(m) \geq 5$. Let y be the $\omega(m)$th prime, and set $n = \prod_{p \leq y} p$.

Note $n \geq 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310 > 1618.2$.

Also, $n \leq m$ (the primes dividing m are at least as large as those dividing n). Thus

$$\omega(m) = \omega(n) \overset{\text{by Lemma}}{=} \frac{\log n}{\log\log n} + O\left(\frac{\log n}{(\log\log n)^2}\right)$$

$$\leq \frac{\log m}{\log\log m} + O\left(\frac{\log m}{(\log\log m)^2}\right). \quad //$$

Exercise: note if $n = \prod_{p \leq y} p$ then $\frac{n}{\phi(n)} = \prod_{p \leq y}\left(1 - \frac{1}{p}\right)^{-1} = e^{C_0} \log y + O(1)$. (and $\log n \sim y$).

A similar proof gives:

Theorem 2.9: For $n \geq 3$,

$$\phi(n) \geq \frac{n}{e^{C_0} \log\log n}\left(1 + O\left(\frac{1}{\log\log n}\right)\right).$$

(In particular, $\phi(n) \gg_\varepsilon n^{1-\varepsilon}$ for all $\varepsilon > 0$.)

Next goal: maximal order of $d(n)$.

Trivial: $d(n) \leq n$.

Almost trivial: Pair divisors d & n/d, so $d \leq \sqrt{n} \Rightarrow d(n) \leq 2\sqrt{n}$.

Example goal: Show there exists $C > 0$ such that $d(n) \leq C n^{1/4}$ for all $n \geq 1$; determine the best possible value for C.

Notation: $p^\alpha \| n$ means $p^\alpha | n$ and $p^{\alpha+1} \nmid n$.

If f is multiplicative, then $f(n) = \prod_{p^\alpha \| n} f(p^\alpha)$.

Solution: Since $d(n)/n^{1/4}$ is multiplicative, we have

$$\frac{d(n)}{n^{1/4}} = \prod_{p^\alpha \| n} \frac{d(p^\alpha)}{(p^\alpha)^{1/4}} = \prod_{p^\alpha \| n} \frac{\alpha+1}{p^{\alpha/4}}.$$

For a given p, which of $\left\{1, \frac{2}{p^{1/4}}, \frac{3}{p^{2/4}}, \frac{4}{p^{3/4}}, \dots\right\}$ is largest? Consider consecutive ratios:

$$\frac{\text{value}(\alpha)}{\text{value}(\alpha-1)} = \frac{(\alpha+1)/p^{(\alpha+1)/4}}{\alpha/p^{\alpha/4}} = \left(1+\frac{1}{\alpha}\right)\frac{1}{p^{1/4}}.$$

- If $p > 2^4 = 16$, then $\left(1+\frac{1}{1}\right)\frac{1}{p^{1/4}} < 1$, so $1 > \frac{2}{p^{1/4}} > \frac{3}{p^{2/4}} > \dots$.
- If $\left(\frac{3}{2}\right)^4 < p < 2^4$, then $\frac{2}{p^{1/4}} > 1$ and $\frac{2}{p^{1/4}} > \frac{3}{p^{2/4}} > \dots$.

$p = 7, 11, 13$

- Similarly, $\alpha = 2$ maximal for $p = 5$
  $\alpha = 3$ " $p = 3$
  $\alpha = 5$ " $p = 2$.

So if $B = 2^5 3^3 5^2 7 \cdot 11 \cdot 13 = 21{,}621{,}600$, then for $n \geq 1$,

$$\frac{d(n)}{n^{1/4}} \leq \frac{d(B)}{B^{1/4}} = \frac{6}{2^{5/4}} \frac{4}{3^{3/4}} \frac{3}{5^{2/4}} \frac{2}{7^{1/4}} \frac{2}{11^{1/4}} \frac{2}{13^{1/4}} \approx 8.447.$$

Hence $d(n) \leq (\approx 8.447)\, n^{1/4}$ for all $n \geq 1$, with equality when $n = B$. //

This proof generalizes: for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that $d(n)\le C_\varepsilon n^\varepsilon$ for all $n\ge 1$. $(d(n)\ll_\varepsilon n^\varepsilon)$:

Given p, when is $\frac{\alpha+1}{p^{\alpha\varepsilon}}$ maximal?

- Consecutive ratios are $(1+\frac{1}{\alpha})\frac{1}{p^\varepsilon}$, which exceeds 1 when $p^\varepsilon<1+\alpha^{-1}$, or $\alpha<(p^\varepsilon-1)^{-1}$. So set $\alpha_\varepsilon(p)=\lfloor (p^\varepsilon-1)^{-1}\rfloor$.

(Choose $\alpha_\varepsilon(p)=0$ when $p>2^{1/\varepsilon}$, $\alpha_\varepsilon(p)=1$ when $(\frac{3}{2})^{\frac{1}{\varepsilon}}<p<2^{1/\varepsilon}$, etc.) By the argument above, $d(n)\le C(\varepsilon)n^\varepsilon$ where

$$C(\varepsilon)=\prod_{p\le 2^{1/\varepsilon}}\frac{\alpha_\varepsilon(p)+1}{p^{\alpha_\varepsilon(p)\varepsilon}}.$$

Let's bound $C(\varepsilon)$! Note: $\alpha_\varepsilon(p)\le (p^\varepsilon-1)^{-1}$

$$\le (2^\varepsilon-1)^{-1}$$

$$\sim \frac{1}{\varepsilon\log 2}\ll \varepsilon^{-1}$$

as $\varepsilon\to 0^+$.

$$\log C(\varepsilon)=\sum_{p\le 2^{1/\varepsilon}}\left(\log(\alpha_\varepsilon(p)+1)-\varepsilon\alpha_\varepsilon(p)\log p\right)$$

$$\le\sum_{p\le 2^{1/\varepsilon}}(\log 2-\varepsilon\log p)+O\Big(\sum_{p\le(\frac{3}{2})^{1/\varepsilon}}\log\tfrac{1}{\varepsilon}\Big)$$

$$=\underset{\varepsilon\cdot\log(2^{1/\varepsilon})}{\cancel{\log 2}}\cdot\pi(2^{\frac{1}{\varepsilon}})-\varepsilon\,\theta(2^{1/\varepsilon})+O\big(\log\tfrac{1}{\varepsilon}\cdot\pi((\tfrac{3}{2})^{\frac{1}{\varepsilon}})\big)$$

From Group Work #3/#4, $\theta(x)=\pi(x)\log x+O(\frac{x}{\log x})$, so with $x=2^{\frac{1}{\varepsilon}}$,

$$\log C(\varepsilon)\ll\varepsilon\left(\frac{2^{1/\varepsilon}}{\log 2^{1/\varepsilon}}\right)+\log\tfrac{1}{\varepsilon}\cdot(\tfrac{3}{2})^{\frac{1}{\varepsilon}}$$

$$\ll \varepsilon^2 2^{1/\varepsilon}.$$

Hence $\log d(n)\le\log C(\varepsilon)+\log(n^\varepsilon)$

$$\ll\varepsilon^2 2^{1/\varepsilon}+\varepsilon\log n,$$ for any $\varepsilon>0$.

We choose ε so that $2^{1/\varepsilon}=\log n$ (as a heuristic to roughly minimize the sum): $\varepsilon=\frac{\log 2}{\log\log n}$.

Theorem 2.11: For $n\ge 3$,

$$\log d(n)\le\frac{\log n}{\log\log n}\left(\log 2+O\Big(\frac{1}{\log\log n}\Big)\right).$$

$$\Rightarrow d(n)\le n^{(\log 2)/\log\log n+O(\cdots)}.$$

Theorem 2.11: For $n \geq 3$,

$$\log d(n) \underset{(*)}{\leq} \frac{\log n}{\log\log n}\left(\log 2 + O\left(\frac{1}{\log\log n}\right)\right).$$

$$\Rightarrow d(n) \leq n^{(\log 2)/\log\log n + O(\cdots)}$$

It turns out that $(*)$ is the best possible upper bound, since $d(n) \geq 2^{\omega(n)}$, and so $\log d(n) \geq \omega(n) \log 2$, — compare Theorem 2.10. //

Enter complex analysis ... (Chapter 5).

Given a sequence $a_1, a_2, \ldots$, define

$$A(x) = \sum_{n \leq x} a_n \quad \text{and} \quad \alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

We saw in Theorem 1.3 that, for $\sigma > \max\{0, \sigma_c\}$,

$$\alpha(s) = s \underbrace{\int_0^\infty A(x) x^{-s-1} dx}.$$

This is called the Mellin transform of $A(x)$.

Aside: In fact, (Mellin transform of $f(x)$)(s) is the same as (Fourier transform of $f(e^{-x})$)$(-is)$.

We want an inverse Mellin transform, writing $A(x)$ in terms of $\alpha(s)$. Heuristic:

- $$\frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{y^s}{s}\, ds = \begin{cases} 1, & \text{if } y > 1, \\ \frac{1}{2}, & \text{if } y = 1, \\ 0, & \text{if } 0 < y < 1. \end{cases}$$

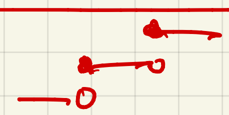
- $$\frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \alpha(s) \frac{x^s}{s}\, ds = \frac{1}{2\pi i}\int \sum_{n=1}^{\infty} a_n n^{-s} \frac{x^s}{s}\, ds$$

$$\text{“}=\text{”} \sum_{n=1}^{\infty} a_n \cdot \frac{1}{2\pi i}\int \frac{(x/n)^s}{s}\, ds$$

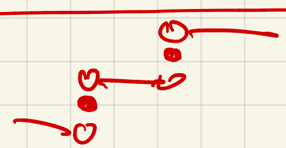
$$= \sum_{n=1}^{\infty} a_n \cdot \begin{cases} 1, & \text{if } n < x, \\ \frac{1}{2}, & \text{if } n = x, \\ 0, & \text{if } n > x \end{cases}$$

$$= \sum_{n \leq x}{}' a_n,$$ where $'$ means a_x is counted w/ weight $\frac{1}{2}$ if $x \in \mathbb{N}$.

$A(x)$ -



$A_0(x)$



Outcome of heuristic:

if $A_0(x) = \sum_{n \le x}' a_n$, then

$$A_0(x) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \alpha(s) \frac{x^s}{s} \, ds$$

– inverse Mellin transform.

Need:

- rigorous proof
- qualitative version, involving

$$\int_{\sigma_0 - iT}^{\sigma_0 + iT}.$$

Definition: Define the sine integral

$$\mathrm{si}(y) = -\int_y^\infty \frac{\sin u}{u} \, du.$$

In particular, $\mathrm{si}(y) \ll 1.$