Thursday, February 13

Group Work #5 today

RECALL

Definition: Define the sine integral

$$\mathrm{si}(y) = -\int_y^\infty \frac{\sin u}{u}\,du.$$

Lemma: For $y>0$, $\mathrm{si}(y) \ll 1/y$.

Proof: $\mathrm{si}(y) = \int_y^\infty \frac{-\sin u}{u}\,du$

$$= \frac{\cos u}{u}\Big|_y^\infty + \int_y^\infty \frac{\cos u}{u^2}\,du;$$ now

just use $\cos u \ll 1$. //

Corollary: For $y>0$, $\mathrm{si}(y) \ll \min\{1, 1/y\}$.

So! In today's group work you'll prove:

Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, and let

$$A_0(x) = \sum_{n \le x}{}' a_n = \sum_{n<x} a_n + \frac{1}{2} a_x \text{ if } x \in \mathbb{N}.$$

If $\sigma_0 > \max\{0, \sigma_a\}$ then, for $x, T \ge 1$:

$$(*) \quad A_0(x) = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s} \, ds$$

$$+ \frac{1}{\pi} \sum_{\frac{x}{2} < n < x} a_n \operatorname{si}\left(T \log \frac{x}{n}\right)$$

$$- \frac{1}{\pi} \sum_{x < n < 2x} a_n \operatorname{si}\left(T \log \frac{n}{x}\right)$$

$$+ O\left(\frac{(4x)^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0}} \right).$$

Some simplifications:

- Suppose $x < n < 2x$. Then $0 < \frac{n-x}{x} < 1$, and so

$$\log \frac{n}{x} = \log\left(1 + \frac{n-x}{x}\right) \asymp \frac{n-x}{x} \quad \left(\begin{array}{l}\text{linear} \\ \text{approx.}\end{array}\right)$$

Hence $\operatorname{si}\left(T \log \frac{n}{x}\right) \ll \min\left\{1, \frac{1}{T \log n/x}\right\}$

$$\ll \min\left\{1, \frac{x}{T(n-x)}\right\}.$$

- Suppose $\frac{x}{2} < n < x$. Then $0 < \frac{x-n}{x} < \frac{1}{2}$, and so

$$\log \frac{x}{n} = \log\left(1 - \frac{x-n}{x}\right)^{-1} \asymp \frac{x-n}{x}. \quad \text{Hence}$$

$$\operatorname{si}\left(T \log \frac{x}{n}\right) \ll \min\left\{1, \frac{1}{T \log \frac{x}{n}}\right\} \ll \min\left\{1, \frac{x}{T(x-n)}\right\}.$$

Then $(*)$ becomes

$$A_0(x) = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s} \, ds + O\left(\sum_{\substack{\frac{x}{2} < n < 2x \\ n \ne x}} |a_n| \min\left\{1, \frac{x}{T|n-x|}\right\} + \frac{(4x)^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0}} \right).$$

$$\boxed{\text{Perron's formula}}$$

$$A_0(x) = \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} \alpha(s)\frac{x^s}{s}\,ds \;+$$

$$+\,O\left(\sum_{\substack{\frac{x}{2}<n<2x\\ n\neq x}} |a_n| \min\left\{1, \frac{x}{T|n-x|}\right\} + \frac{(4x)^{\sigma_0}}{T}\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0}}\right)$$

Note: if we take $\lim_{T\to\infty}$ of both sides,

$$A_0(x) = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \alpha(s)\frac{x^s}{s}\,ds.$$

Plan: apply Perron's formula to $a_n = \Lambda(n)$, so that $A_0(x) = \psi_0(x) = {\sum_{n\le x}}' \Lambda(n)$ and

$$\sum_{n=1}^{\infty} \Lambda(n) n^{-s} = -\frac{\zeta'}{\zeta}(s).$$

Use residue calculus to evaluate the main term.

Need: better understanding of zeros of $\zeta(s)$.

Estimating contour integrals:

$$\left|\int_C f(s)\,ds\right| \le \int_C |f(s)|\,|ds|$$

special cases:

- $|ds| = d\sigma$ if $C = \rightarrow$
- $|ds| = dt$ if $C = \uparrow$.