Tuesday, February 25

Suggested Problems #3 posted Thursday

Recall: $\sum_{n=1}^{\infty} \Lambda(n) n^{-s} = -\frac{\zeta'}{\zeta}(s)$. So we saw that

$$\psi_0(x) = \sum_{n \le x}{}' \Lambda(n) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \left(-\frac{\zeta'}{\zeta}(s)\right) \frac{x^s}{s}\, ds$$

For any $\sigma_0 > 1$. We're going to use residue calculus to get the prime number theorem from this: $\psi(x) \sim x$. To do this, we need to understand better the zeros of $\zeta(s)$.

Lemma 6.5: For $\sigma > 1$ and $t \neq 0$,

$$\mathrm{Re}\left(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma+it) - \frac{\zeta'}{\zeta}(\sigma+2it)\right) \ge 0$$

and $$|\zeta(\sigma)^3 \zeta(\sigma+it)^4 \zeta(\sigma+2it)| \ge 1.$$

Proof: We use the fun inequality

$$3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2 \ge 0.$$

Then
$$\mathrm{Re}\left(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma+it) - \frac{\zeta'}{\zeta}(\sigma+2it)\right)$$
$$= \mathrm{Re}\left(3\sum_{n=1}^{\infty} \Lambda(n) n^{-\sigma} + 4\sum_{n=1}^{\infty} \Lambda(n) n^{-\sigma-it} + \sum_{n=1}^{\infty} \Lambda(n) n^{-\sigma-2it}\right)$$
$$= \sum_{n=1}^{\infty} \Lambda(n) n^{-\sigma}\, \mathrm{Re}\left(3 + 4n^{-it} + n^{-2it}\right)$$
$$= \sum_{n=1}^{\infty} \Lambda(n) n^{-\sigma}\left(3 + 4\cos(t\log n) + \cos(2t\log n)\right) \ge 0.$$

(since $n^{-it} = e^{-it\log n}$).

Next: for $y \in \mathbb{R}$,
$$\int_{\sigma}^{\infty} -\frac{\zeta'}{\zeta}(x+iy)\, dx = -\log \zeta(x+iy) \Big|_{\sigma}^{\infty}$$
$$= 0 - (-\log \zeta(\sigma+iy))$$

and so
$$\int_{\sigma}^{\infty} \mathrm{Re}\left(-\frac{\zeta'}{\zeta}(x+iy)\right) dx = \mathrm{Re} \log \zeta(\sigma+iy)$$
$$= \log|\zeta(\sigma+iy)|.$$

Thus $\log|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)|$

$$= \operatorname{Re}\log\left(\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)\right)$$

$$= \operatorname{Re}\left(3\log\zeta(\sigma)+4\log\zeta(\sigma+it)+1\log\zeta(\sigma+2it)\right)$$

$$= \int_\sigma^\infty \operatorname{Re}\left(-3\frac{\zeta'}{\zeta}(x)-4\frac{\zeta'}{\zeta}(x+it)-\frac{\zeta'}{\zeta}(x+2it)\right)dx$$

$$\geq 0,$$

and thus $|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)|\geq 1$.

(for $\sigma>1$, $t\neq 0$) //

Note: this inequality immediately implies that $\zeta(\sigma+it)\neq 0$ for $\sigma>1$. (We also saw this from the absolutely convergent Euler product for $\zeta(s)$.)

New conclusion: $\zeta(s)\neq 0$ for $\sigma=1$.

Proof: Of course $\zeta(1)\neq 0$. Suppose $\zeta(1+it)=0$, $t\neq 0$.

Then $f(s)=\zeta(s)^3\zeta(s+it)^4\zeta(s+2it)$, near $s=1$, looks like (triple pole) $\times$ (at least quadruple zero) $\times$ (not a pole) and thus $f(s)$ vanishes at $s=1$.

But then $\lim_{\sigma\to 1^+}|f(\sigma)|\geq 1$ by previous lemma

but $\lim_{\sigma\to 1^+}|f(\sigma)|=f(1)=0$, contradiction //

It turns out that "$\zeta(s)\neq 0$ for $\sigma=1$" is actually equivalent to the prime number theorem ("$\psi(x)\sim x$" or "$\pi(x)\sim\frac{x}{\log x}$").

But this equivalence is delicate; and also we want a stronger statement $\psi(x)=x+O(\text{something specific})$.

For this we need $\zeta(s)\neq 0$ in some (specific) open neighbourhood of $\{\sigma=1\}$.

We need some tools from medium-core complex analysis.

- Suppose $f(z)$ is analytic on $\{|z| \le R\}$.

Define the Blaschke product

$$g(z) = f(z) \prod_{\substack{w \in \mathbb{C} \\ |w| \le R \\ f(w)=0}} \frac{R^2 - z\bar{w}}{R(z-w)}.$$

(If w is a multiple zero of f, it appears multiple times in the product.) Check:

- $g(z)$ is also analytic on $\{|z| \le R\}$
- $|g(z)| = |f(z)|$ on $\{|z| = R\}$.
- if $|f(z)| \le M$ for $|z| \le R$, then also $|g(z)| \le M$ there (maximum modulus principle).

Exercise (Lemma 6.1):

Suppose $f(z)$ is analytic on $\{|z| \le R\}$ and satisfies $|f(z)| \le M$ there, and $f(0) \ne 0$. For $0 < r < R$,

$$\#\{|w| \le r : f(w) = 0\} \le \frac{\log(M/|f(0)|)}{\log(R/r)}.$$

Hint: get an upper bound on $|f(0)|$ using the Blaschke product. (Relevant words: "Jensen's formula")

Suppose h is analytic on $\{|z| \le R\}$ and satisfies $|h(z)| \le M$ there, and $h(0) = 0$. Recall by Schwarz's Lemma,

$$|h(z)| \le M\frac{|z|}{R},$$

and so

$$|h(z)| \le M\frac{r}{R} \text{ for all } |z| \le r.$$

__Borel - Carathéodory Lemma__ [Lemma 6.2]:

Suppose h is analytic on $\{|z| \leq R\}$ and satisfies $\operatorname{Re} h(z) \leq M$ there, and $h(0) = 0$. Then for any $0 < r < R$,

$$|h(z)| \leq \frac{2Mr}{R-r} \quad \text{and} \quad |h'(z)| \leq \frac{2Mr}{(R-r)^2}.$$

__Sketch:__ For all $k \geq 0$, by the Cauchy integral formula,

$$\frac{h^{(k)}(0)}{k!} = \frac{1}{2\pi i} \oint_{|z|=R} \frac{h(z)}{z^{k+1}}\, dz.$$

If we had $|h(z)| \leq M$, we would get

$$\left|\frac{h^{(k)}(0)}{k!}\right| \leq \frac{1}{2\pi} \oint_{|z|=R} \frac{|h(z)|}{|z|^{k+1}} |dz|$$

$$\leq \frac{1}{2\pi} \cdot 2\pi R \cdot \frac{M}{R^{k+1}} = \frac{M}{R^k}.$$

A less obvious argument still gives: if $\operatorname{Re} h(z) \leq M$ then $\left|\frac{h^{(k)}(0)}{k!}\right| \leq \frac{2M}{R^k}$.

(See book; trick - when $|z| = R$, we have $\operatorname{Re} z = \frac{1}{2}(z + \bar{z}) = \frac{1}{2}\left(z + \frac{R^2}{z}\right)$.)

Then for $|z| \leq r$, we bound the power series

$$|h(z)| = \left|\sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} z^k\right|$$

$$\leq \sum_{k=1}^{\infty} \left|\frac{h^{(k)}(0)}{k!}\right| |z|^k \leq \sum_{k=1}^{\infty} \frac{2M}{R^k} r^k$$

$$= \frac{2Mr}{R-r}.$$

Similar argument for

$$|h'(z)| = \left|\sum_{k=1}^{\infty} \frac{h^{(k)}(0)}{k!} k z^{k-1}\right| \cdots$$

//

Motivating observation:

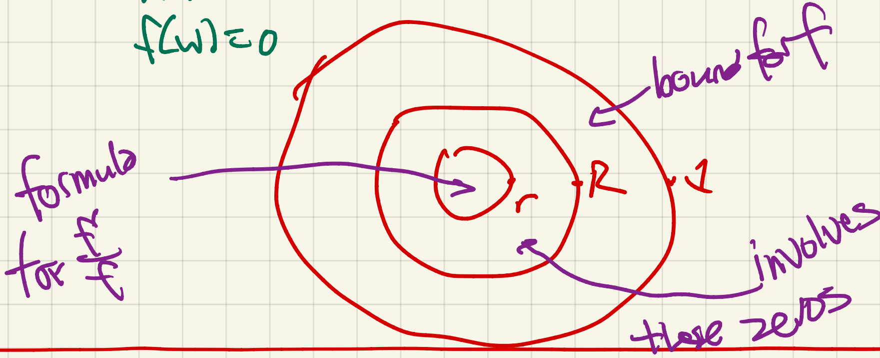
If $P(z)$ is a polynomial, then

$$\frac{P'}{P}(z) = \sum_{\substack{w\in\mathbb{C}\\ P(w)=0}} \frac{1}{z-w}.$$

Can we do something similar for arbitrary analytic functions?

Lemma 6.3: Suppose f is analytic and satisfies $|f(z)|\le M$ on the big disk $\{|z|\le 1\}$, and $f(0)\neq 0$. Then, on the small disk $\{|z|\le r\}$,

$$\frac{f'}{f}(z) = \sum_{\substack{|w|\le R\\ f(w)=0}} \frac{1}{z-w} + O_{r,R}\left(\log\frac{M}{|f(0)|}\right).$$



$0<r<R<1.$

Idea of proof: Let $g(z) = f(z)\prod\limits_{\substack{|w|<R\\ f(w)=0}} \frac{R^2 - z\bar{w}}{R(z-w)}$, and $h(z) = \log\frac{g(z)}{g(0)}$, so that

$$\operatorname{Re} h(z) = \log\left|\frac{g(z)}{g(0)}\right|$$

$$= \log|g(z)| - \log\Big|f(0)\prod_{\substack{|w|<R\\ f(w)=0}} \frac{R}{w}\Big|$$

$$\le \log M - \log|f(0)|.$$

Apply Borel–Carathéodory to $h(z)$ to bound $h'(z) = \frac{g'}{g}(z)$

$$= \frac{f'}{f}(z) - \sum_w \frac{1}{z-w} + \sum_w \frac{1}{z-R^2/\bar{w}}$$

and bound the second sum. //

Next we'll apply Lemma 6.3 to $\zeta(s)$.

<u>Notation:</u> given $s \in \mathbb{C}$, define

$$\tau = |t| + 4 = |\operatorname{Im} s| + 4.$$

<u>Lemma 6.4:</u> Suppose $\frac{5}{6} \leq \sigma \leq 2$ and $|t| \geq \frac{7}{8}$. Then

$$\frac{\zeta'}{\zeta}(s) = \sum_{\substack{\rho \\ |\rho - (\frac{3}{2}+it)| \leq 5/6 \\ \zeta(\rho)=0}} \frac{1}{s-\rho} + O(\log \tau).$$