Thursday, February 27

Suggested Problems #3 posted today

Group Work #6 next Tuesday

RECALL $[0<r<R<1]$

Lemma 6.3: Suppose f is analytic and satisfies $|f(z)| \le M$ on the big disk $\{|z|\le 1\}$, and $f(0)\neq 0$. Then, on the small disk $\{|z|\le r\}$,

$$\frac{f'}{f}(z) = \sum_{\substack{|w|\le R \\ f(w)=0}} \frac{1}{z-w} + O_{r,R}\left(\log \frac{M}{|f(0)|}\right).$$

← sum over zeros in medium disk

Notation: given $s\in\mathbb{C}$, define

$$\tau = |t|+4 = |\operatorname{Im} s| + 4.$$

STATED ON TUESDAY

Lemma 6.4: Suppose $\frac{5}{6}\le\sigma\le 2$ and $|t|\ge\frac{7}{8}$. Then

$$\frac{\zeta'}{\zeta}(s) = \sum_{\substack{|\rho-(\frac{3}{2}+it)|\le \frac{5}{6} \\ \zeta(\rho)=0}} \frac{1}{s-\rho} + O(\log\tau).$$

Proof: We apply Lemma 6.3 to $f(z) = \zeta(z+\frac{3}{2}+it)$, with $r=\frac{2}{3}$ and $R=\frac{5}{6}$. The main term appears immediately.

- $|f(0)|^{-1} = |\zeta(\frac{3}{2}+it)|^{-1} = \prod_p |1-p^{-\frac{3}{2}-it}|$

$$\le \prod_p (1+p^{-3/2}) = \frac{1}{\zeta(3/2)} \ll 1.$$

- Corollary 1.17 implies $\zeta(s) \ll \tau^{1/2}$ for $\sigma\ge\frac{1}{2}$, so $f(z)\ll\tau^{1/2}$ for $|z|\le 1$.

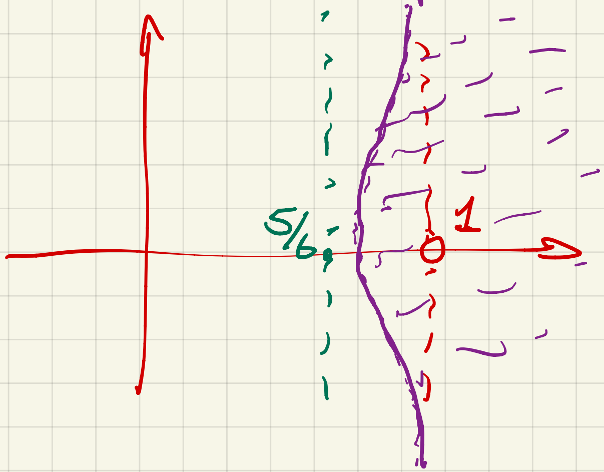
Hence $O_{r,R}\left(\log\frac{M}{|f(0)|}\right) = O_{\frac{2}{3},\frac{5}{6}}\left(\log\frac{\tau^{1/2}}{1}\right)$

$$= O(\tfrac{1}{2}\log\tau) = O(\log\tau). \quad //$$

We can now establish the "classical zero-free region" for $\zeta(s)$.

Theorem 6.6 (de la Vallée-Poussin, 1899)

There exists an absolute constant $0<c<1$ such that $\zeta(s) \neq 0$ for $\sigma \geq 1 - \frac{c}{\log \tau}$.



Proof: On Suggested Problems #4, you'll prove that $\zeta(s) \neq 0$ for $0 \leq \sigma \leq 1$, $|t| \leq 6$.

Let $\rho_0 = \beta_0 + i\gamma_0$ be a zero of $\zeta(s)$ with $|\gamma_0| \geq 6 > \frac{7}{8}$ and $\frac{5}{6} \leq \beta_0 < 1$.

For any $\delta > 0$, we proved

$$0 \leq \mathrm{Re}\left(-3\frac{\zeta'}{\zeta}(1+\delta) - 4\frac{\zeta'}{\zeta}(1+\delta+i\gamma_0) - \frac{\zeta'}{\zeta}(1+\delta+2i\gamma_0)\right).$$

(*)

- $\frac{\zeta'}{\zeta}$ has a simple pole of residue -1 at $s=1$, so $\frac{\zeta'}{\zeta}(s) \approx \frac{-1}{s-1}$ nearby. More specifically (Corollary 1.3), $-\frac{\zeta'}{\zeta}(1+\delta) = \frac{1}{\delta} + O(1)$.
- If $\zeta(\rho)=0$ then $\mathrm{Re}\,\rho < 1$, so $\mathrm{Re}\,\frac{-1}{s-\rho} < 0$ for $\sigma > 1$. Hence we can apply Lemma 6.4 and throw away as many summands as we want to get an upper:

$$\mathrm{Re}\left(-\frac{\zeta'}{\zeta}(1+\delta+i\gamma_0)\right) = \mathrm{Re}\left(\sum_{\substack{|\rho - (\frac{3}{2}+i\gamma_0)| \leq \frac{5}{6} \\ \zeta(\rho)=0}} -\frac{1}{1+\delta+i\gamma_0-\rho} + O(\log \tau_0)\right)$$

$\tau_0 = |\gamma_0| + 4$

$$\leq \mathrm{Re}\left(-\frac{1}{1+\delta+i\gamma_0-(\beta_0+i\gamma_0)} + O(\log \tau_0)\right)$$

$$= -\frac{1}{1+\delta-\beta_0} + O(\log \tau_0).$$

- Similarly, but throwing all summands away, $-\frac{\zeta'}{\zeta}(1+\delta+2i\gamma_0) \leq 0 + O(\log \tau_0)$.

Therefore $(*)$ implies

$$0 \le \frac{3}{\delta} \cancel{+O(1)} - \frac{4}{1+\delta-\beta_0} + O(\log \tau_0).$$

Choose $b>0$ so that $\frac{3}{\delta} - \frac{4}{1+\delta-\beta_0} + b\log\tau_0 \ge 0$.
?

Now choose $\delta = \frac{1}{2b\log\tau_0}$:

$$\frac{3}{\delta} + b\log\tau_0 \ge \frac{4}{1+\delta-\beta_0}$$

$$7b\log\tau_0 \ge \frac{4}{1-\beta_0+1/2b\log\tau_0}$$

$$1-\beta_0+\frac{1}{2b\log\tau_0} \ge \frac{4}{7b\log\tau_0}$$

$$1-\beta_0 \ge \frac{1}{14}\,\frac{1}{b\log\tau_0}.$$

This establishes the zero-free region with

$c = \frac{1}{14b}$. //

Recall the overall plan:

- $\psi_0(x) = \frac{1}{2\pi i}\int -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds$
- pull the contour to the left
- no residues except at $s=1$
  (since we now have a zero-free region)
- estimate the new contour integrals.

Theorem 6.7: Let $c>0$ be such that $\zeta(s)\neq 0$ for $\sigma \ge 1-\frac{c}{\log\tau}$. Uniformly for $\sigma \ge 1-\frac{c/2}{\log\tau}$:

- $\frac{\zeta'}{\zeta}(s) \ll \log\tau$
- $|\log\zeta(s)| \le \log\log\tau + O(1)$
- $\zeta(s) \ll \log\tau$ and $\frac{1}{\zeta(s)} \ll \log\tau.$

Sketch of proof: Compare $\frac{\zeta'}{\zeta}(s)$ to $\frac{\zeta'}{\zeta}(s_1)$ with $s_1 = 1+\frac{1}{\log\tau}+it$:

$$\left|\frac{\zeta'}{\zeta}(s_1)\right| = \left|\sum_{n=1}^{\infty}\Lambda(n)n^{-s_1}\right|$$
$$\leq \sum_{n=1}^{\infty}\Lambda(n)\,n^{-1-\frac{1}{\log T}}$$
$$= -\frac{\zeta'}{\zeta}\left(1+\frac{1}{\log T}\right) = \log T + O(1).$$

Then use Lemma 6.4 and bound

$$\frac{\zeta'}{\zeta}(s) - \frac{\zeta'}{\zeta}(s_1) = \sum_{\text{nearby } \rho}\left(\frac{1}{s-\rho} - \frac{1}{s_1-\rho}\right) + O(\log T) \ldots$$

- Similarly, $|\log\zeta(s_1)| \leq \log\zeta\left(1+\frac{1}{\log T}\right)$,

and $\log\zeta(s) - \log\zeta(s_1) = \int_{s_1}^{s}\frac{\zeta'}{\zeta}(w)\,dw$;

estimate using previous bound.

- $\log|\zeta(s)| = \operatorname{Re}\log\zeta(s) \leq |\log\zeta(s)|$
$\leq \log\log T + O(1)$;

exponentiate to get

$$|\zeta(s)| \leq e^{\log\log T}e^{O(1)} \ll \log T. \quad //$$

Setup of the proof of the Prime Number Theorem $\psi(x) \sim x$.

Start from "Perron's formula", applied to $a_n = \Lambda(n)$, so that $\sum'_{n\leq x} a_n = \psi_0(x)$ and $\sum_{n=1}^{\infty} a_n n^{-s} = -\frac{\zeta'}{\zeta}(s)$: for any $\sigma_0 > 1$,

$$(**)\quad \psi_0(x) = \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0+iT}\left(-\frac{\zeta'}{\zeta}(s)\right)\frac{x^s}{s}\,ds$$
$$+ O\left(\sum_{\substack{\frac{x}{2}<n<2x \\ n\neq x}}\Lambda(n)\min\left\{1, \frac{x}{T|x-n|}\right\} + \frac{(4x)^{\sigma_0}}{T}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma_0}}\right).$$

Simplifications:
- Assume $x > T > e$.
- $\Lambda(n) \ll \log x$ for all $n < 2x$. In particular, $\psi_0(x) = \psi(x) - \frac{1}{2}\text{"}\Lambda(x)\text{"} = \psi(x) + O(\log x)$.

Thus $\displaystyle\sum_{\substack{\frac{x}{2}<n<2x \\ n\neq x}} \Lambda(n)\min\left\{1, \frac{x}{T|x-n|}\right\}$

$$\ll (\log x)\Big(\underbrace{1+1}_{n=\lfloor x\rfloor,\lceil x\rceil} + \underbrace{\sum_{1\le k\le x}\frac{x}{Tk}}_{n=x+k} + \underbrace{\sum_{1\le k\le \frac{x}{2}}\frac{x}{Tk}}_{n=x-k}\Big)$$

$$\ll (\log x)\left(1+\frac{x}{T}\log x\right) \ll \frac{x}{T}\log^2 x.$$

- We must choose $\sigma_0 > \max\{0, \sigma_a\} = 1$.

We choose $\sigma_0 = 1+\frac{1}{\log x} \le 2$. Note that $4^{\sigma_0}\le 16$ and $x^{\sigma_0} = ex$ (exactly!). So

$$\frac{(4x)^{\sigma_0}}{T}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma_0}} \ll \frac{x}{T}\left(-\frac{\zeta'}{\zeta}\left(1+\frac{1}{\log x}\right)\right)$$

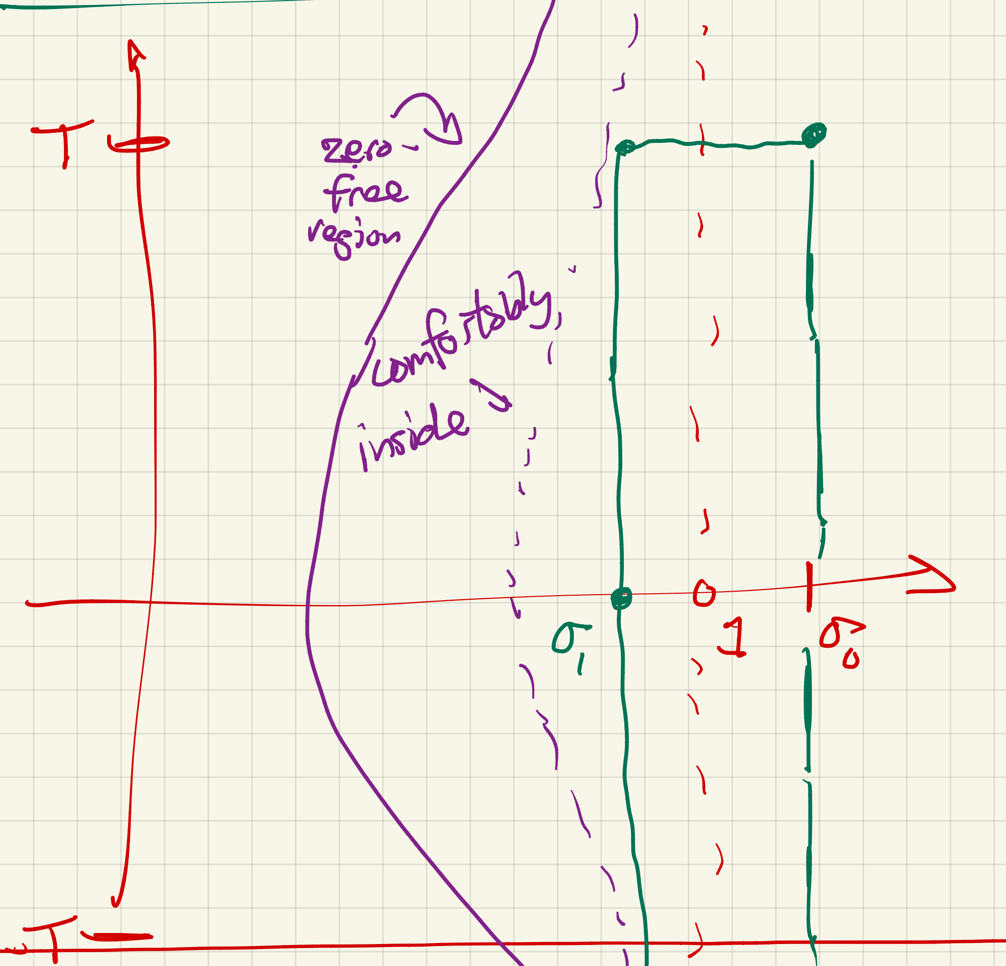
$$\ll \frac{x}{T}\cdot\frac{1}{1/\log x}.$$

Therefore (**) becomes

$$\psi_0(x) = \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds \quad (***)$$
$$+\,O\left(\frac{x}{T}\log^2 x\right)$$

where $\sigma_0 = 1+1/\log x$. $(x>T>e)$.

Setup of contour integration



$$\sigma_1 = 1-\frac{c/2}{\log(T+4)}.$$

Choose $\sigma_0 = 1 + \frac{1}{\log x}$ and $\sigma_1 = 1 - \frac{c/2}{\log(T+4)}$

(c = constant from zero-free region). Consider

$$\frac{1}{2\pi i} \oint_R -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s}\, ds$$ where $\underline{R}$ is

the rectangle with corners $\sigma_0 \pm iT$ and $\sigma_1 \pm iT$.

- The only singularity of the integrand inside R is the simple pole of $-\frac{\zeta'}{\zeta}(s)$ at $s=1$, with residue 1. Hence the residue of $-\frac{\zeta'}{\zeta}(s) \frac{x^s}{s}$ at $s=1$ is

$$\left. \frac{x^s}{s} \right|_{s=1} = x.$$

~~because~~ simple pole!!!

Stopping point:

$$\frac{1}{2\pi i} \oint_{\underline{R}} -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s}\, ds = x.$$

- estimate top / bottom / left side
- right side of rectangle is $\approx \psi(x)$ by Perron's formula.