Tuesday, February 4

Group Work #3 moved to Thursday

Suggested Problems #2 posted Thursday

Recall the von Mangoldt Lambda-function

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k, \\ 0, & \text{otherwise.} \end{cases}$$

- $L(x) = \sum_{n \le x} \log n = \sum_{d \le x} \Lambda(d) \left\lfloor \frac{x}{d} \right\rfloor.$

- Exercise (compare to $\int_1^x \log t \, dt$):

$$L(x) = \sum_{n \le x} \log n = x \log x - x + O(\log x).$$

Define $L(x) = \sum_{n \le x} \log n.$

Consider $L(x) - 2L\left(\frac{x}{2}\right)$:

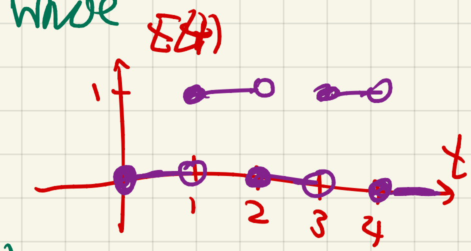
- $= \left(x \log x - x + O(\log x)\right) - 2\left(\frac{x}{2} \log \frac{x}{2} - \frac{x}{2} + O\left(\log \frac{x}{2}\right)\right)$

  $= (\log 2)\, x + O(\log x).$

- $= \sum_{d \le x} \Lambda(d) \left\lfloor \frac{x}{d} \right\rfloor - 2 \sum_{\cancel{d \le x/2}\ d \le x} \Lambda(d) \left\lfloor \frac{x/2}{d} \right\rfloor$

  $= \sum_{d \le x} \Lambda(d)\, E\left(\frac{x}{d}\right)$, where

$$E(t) = \lfloor t \rfloor - 2 \lfloor t/2 \rfloor$$



(psi)

Define $\psi(x) = \sum_{d \le x} \Lambda(d).$

Then

$$\underbrace{\sum_{\frac{x}{2} < d \le x} \Lambda(d)}_{\psi(x) - \psi(x/2)} \le \sum_{d \le x} \Lambda(d)\, E\left(\frac{x}{d}\right) \le \sum_{d \le x} \Lambda(d) = \psi(x)$$

Summary: $L(x) - 2L(\frac{x}{2}) = (\log 2)x + O(\log x)$

and $\psi(x) - \psi(\frac{x}{2}) \leq L(x) - 2L(\frac{x}{2}) \leq \psi(x)$.

Consequences:

- $\psi(x) \geq (\log 2)x + O(\log x)$.
- $\psi(x) = \left(\psi(x) - \psi(\frac{x}{2})\right) + \left(\psi(\frac{x}{2}) - \psi(\frac{x}{4})\right) + \cdots$

$$\leq (\log 2)x + O(\log x) + (\log 2)(\frac{x}{2}) + O(\log \frac{x}{2}) + \cdots$$

$$\leq (2\log 2)x + O(\log_2 x \cdot \log x)$$

$$= (2\log 2)x + O(\log^2 x).$$

Notation: $f(x) \asymp g(x)$ means $f(x) \ll g(x)$ and $g(x) \ll f(x)$.

TeX: \asymp $\asymp$

So $\psi(x) \asymp x$.

Method is due to Chebyshev!

We used $L(x) - 2L(\frac{x}{2})$; he used

$L(x) - L(\frac{x}{2}) - L(\frac{x}{3}) - L(\frac{x}{5}) + L(\frac{x}{30})$

to prove $0.9212x + O(\log x) \leq \psi(x) \leq 1.1056x + O(\log^2 x)$.

In particular,

$$\psi(2x) - \psi(x) \geq 0.9212 \cdot 2x + O(\log x) - 1.1056x + O(\log^2 x)$$

$$= 0.3368x + O(\log^2 x) > 0$$

when x is sufficiently large; so there exists a prime power between x and $2x$ when x is large.

Minor modification establishes

Bertrand's Postulate: When $x \geq 2$, there's always a prime in $[x, 2x]$.

Mertens's theorems (Theorem 2.7) For $x \geq 2$:

(a) $\sum_{n \leq x} \Lambda(n)/n = \log x + O(1)$;

(b) $\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1)$;

(d) $\sum_{p \leq x} \frac{1}{p} = \log\log x + b + O\left(\frac{1}{\log x}\right)$

(e) $\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} = e^C \log x + O(1)$.

for some constants b, C. (d) $\Rightarrow \sum_p \frac{1}{p}$ diverges.

Proof:

(a) $x \log x + O(x) = L(x) = \sum_{d \leq x} \Lambda(d) \left\lfloor \frac{x}{d} \right\rfloor$

$$= \sum_{d \leq x} \Lambda(d)\left(\frac{x}{d} + O(1)\right)$$

$$= x \sum_{d \leq x} \frac{\Lambda(d)}{d} + O(\psi(x)).$$

Dividing by x,

$$\log x + O(1) = \sum_{d \leq x} \frac{\Lambda(d)}{d} + \underbrace{O\left(\frac{\psi(x)}{x}\right)}_{= O(1)}. \checkmark$$

(b) Suffices to show

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} - \sum_{p \leq x} \frac{\log p}{p} = O(1).$$

But this difference equals

$$\sum_{\substack{p^r \leq x \\ r \geq 2}} \frac{\log p}{p^r} < \sum_{p \leq x} \log p \sum_{r=2}^{\infty} \frac{1}{p^r}$$

$$= \sum_{p \leq x} \log p \cdot \frac{1}{p(p-1)}$$

$$\ll \sum_{p \leq x} \sqrt{p} \cdot \frac{1}{p^2} \ll \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} = O(1).$$

Notation:

$R(x) = \sum_{p \leq x} \frac{\log p}{p} - \log x$, so that

(b) says $R(x) \ll 1$. Also $R(2-) = -\log 2$.

(d) $\displaystyle\sum_{p\le x}\frac{1}{p} = \log\log x + b + O\left(\frac{1}{\log x}\right)$

(e) $\displaystyle\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1} = e^{C}\log x + O(1).$

(d) $$\sum_{p\le x}\frac{1}{p} = \int_{2^-}^{x}\frac{1}{\log u}\, d\left(\sum_{p\le u}\frac{\log p}{p}\right)$$

$$= \int_{2^-}^{x}\frac{1}{\log u}\, d(\log u) + \int_{2^-}^{x}\frac{1}{\log u}\, dR(u)$$

$$= \int_{2^-}^{x}\frac{1}{u\log u}\, du + \frac{R(u)}{\log u}\Big|_{2^-}^{x} - \int_{2^-}^{x} R(u)\, d\left(\frac{1}{\log u}\right)$$

$$= \log\log u\Big|_{2^-}^{x} + \frac{R(x)}{\log x} - \frac{R(2^-)}{\log 2^-} + \int_{2^-}^{x}\frac{R(u)}{u\log^2 u}\, du$$

$$= \log\log x - \log\log 2 + O\left(\frac{1}{\log x}\right) - \frac{-\log 2}{\log 2}$$

$$+ \int_{2}^{\infty}\frac{R(u)}{u\log^2 u}\, du - \int_{x}^{\infty}\frac{R(u)}{u\log^2 u}\, du$$

$$= \log\log x + \underbrace{\left(1 - \log\log 2 + \int_{2}^{\infty}\frac{R(u)}{u\log^2 u}\, du\right)} +$$

$$+ O\left(\frac{1}{\log x} + \int_{x}^{\infty}\frac{1}{u\log^2 u}\, du\right) =$$

$$= \log\log x + \underline{b} + O\left(\frac{1}{\log x} + \frac{1}{\log x}\right). \checkmark$$

Sketch of (e): consider

$$\log\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1} = \sum_{p\le x}\log\left(1-\frac{1}{p}\right)^{-1}$$

$$= \sum_{p\le x}\left(\frac{1}{p} + O\left(\frac{1}{p^2}\right)\right) \ldots$$

We get $\displaystyle\log\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1} = \log\log x + C + O\left(\frac{1}{\log x}\right)$

Exponentiating both sides:

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1} = e^{\log\log x}\cdot e^{C}\cdot \underline{e^{O(1/\log x)}} \checkmark$$

$$= \log x\cdot e^{C}\cdot \underline{\left(1+O\left(\frac{1}{\log x}\right)\right)}.$$

Note: we can show that

$C = C_0$ (Euler's constant) and

$\displaystyle b = C_0 - \sum_{p}\sum_{r=2}^{\infty}\frac{1}{r p^r}.$

Takeaways: $\displaystyle\sum_{p\le x}\frac{1}{p} \sim \log\log x$; $\displaystyle\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1} \sim e^{C_0}\log x.$

Applications to arithmetic functions

Recall: $\omega(n) = \#$ of distinct prime factors of n

$\Omega(n) = \#$ of prime factors of n, counted with multiplicity.

Proposition: The average order of $\omega(n)$ is $\log\log n$. (This means

$$\sum_{n\le x} \omega(n) \sim \sum_{n\le x} \log\log n \sim x\log\log x.)$$

[Exercise]

Proof:
$$\sum_{n\le x} \omega(n) = \sum_{n\le x}\sum_{p\mid n} 1$$
$$= \sum_{p\le x}\sum_{\substack{n\le x\\ p\mid n}} 1 = \sum_{p\le x}\left\lfloor \frac{x}{p}\right\rfloor$$
$$= \sum_{p\le x}\frac{x}{p} + O\Big(\sum_{p\le x} 1\Big)$$
$$= x\left(\log\log x + b + O\left(\tfrac{1}{\log x}\right)\right) + O\Big(\sum_{n\le x} 1\Big)$$
$$= x\log\log x + O(x).$$

Lemma: $\displaystyle\sum_{n\le x}\omega(n)^2 \le x(\log\log x)^2 + O(x\log\log x).$

Proof: Throughout, p and q denote primes.

$$\sum_{n\le x}\omega(n)^2 = \sum_{n\le x}\Big(\sum_{p\mid n} 1\Big)^2 = \sum_{n\le x}\sum_{p\mid n}\sum_{q\mid n} 1$$
$$= \sum_{p\le x}\sum_{q\le x}\#\{n\le x : p\mid n \text{ and } q\mid n\}$$
$$= \sum_{\substack{p\le x\\ q=p}}\left\lfloor\frac{x}{p}\right\rfloor + \sum_{p\le x}\sum_{\substack{q\le x\\ q\ne p}}\left\lfloor\frac{x}{pq}\right\rfloor$$
$$\le \sum_{p\le x}\frac{x}{p} + \sum_{p\le x}\sum_{q\le x} x/pq$$
$$= x\sum_{p\le x}\frac{1}{p} + x\Big(\sum_{p\le x}\frac{1}{p}\Big)^2$$
$$= O(x\log\log x) + x(\log\log x + O(1))^2$$
$$= O(x\log\log x) + x\big((\log\log x)^2 + 2\cdot O(1\cdot\log\log x) + O(1^2)\big). \quad \checkmark$$

<u>Proposition</u>: The "variance" of $\omega(n)$ is $O(\log\log x)$.

<u>Proof/actual statement</u>:

$$\frac{1}{x}\sum_{n\le x}(\omega(n)-\log\log x)^2$$

$$= \frac{1}{x}\sum_{n\le x}\omega(n)^2 - \frac{2}{x}\log\log x\sum_{n\le x}\omega(n) + \frac{1}{x}(\log\log x)^2\boxed{\sum_{n\le x}1}$$

$$\le \frac{1}{x}\left(x(\log\log x)^2 + O(x\log\log x)\right) - \frac{2}{x}\log\log x\cdot(x\log\log x + O(x)) + (\log\log x)^2\frac{\lfloor x\rfloor}{x}$$

$$= \underbrace{\cancel{(\log\log x)^2 - 2(\log\log x)^2 + (\log\log x)^2}}_{\emptyset} + O(\log\log x). \checkmark$$

Compare: $\mathrm{Var}(X) = \mathbb{E}((X-\mathbb{E}(X))^2)$

$$= \mathbb{E}(X^2) - 2\mathbb{E}(X\,\mathbb{E}(X)) + \mathbb{E}(\mathbb{E}(X)^2)$$

$$= \mathbb{E}(X^2) - 2\mathbb{E}(X)\mathbb{E}(X) + \mathbb{E}(X)^2$$

$$= \mathbb{E}(X^2) - \mathbb{E}(X)^2.$$

<u>Claim</u>: For most integers n close to x, $\omega(n)$ is close to $\log\log x$:

The number of $n\le x$ for which $|\omega(n)-\log\log x| > (\log\log x)^{\frac{1}{2}+\varepsilon}$ is:

$$\boxed{\sum_{\substack{n\le x\\ |\omega(n)-\log\log x|>(\log\log x)^{\frac12+\varepsilon}}} 1} \overset{(*)}{\le} \sum_{n\le x}\left(\frac{|\omega(n)-\log\log x|}{(\log\log x)^{\frac12+\varepsilon}}\right)^2$$

$$\le \frac{1}{(\log\log x)^{1+2\varepsilon}}\left[\frac{1}{x}\sum_{n\le x}(\omega(n)-\log\log x)^2\right]x$$

$$\ll \frac{\cancel{x}}{(\log\log x)^{2\varepsilon}} = o(x).$$

$(*)$: Compare to proof of Chebyshev's inequality:

$$\Pr(|X-\mathbb{E}X|^2 > \lambda\,\mathrm{Var}(E)) \le \frac{1}{\lambda^2}.$$