Tuesday, January 14

RECALL FROM THURSDAY:

Three key facts about Riemann-Stieltjes integrals:

$\boxed{\text{I}}$ Let $A(x) = \sum_{n \le x} a_n$. If $f(x)$ is any continuous function, then

$$\int_c^d f(x)\,dA(x) = \sum_{c < n \le d} a_n f(n)$$

Idea of proof: Recall

$$\int_c^d f(x)\,dA(x) = \lim_{m(\underline{x}) \to 0} \sum_{i=1}^{N} f(\xi_i)\left(A(x_i) - A(x_{i-1})\right)$$

Note that

$$A(x_i) - A(x_{i-1}) = \begin{cases} 0, & \text{if } (x_{i-1}, x_i] \cap \mathbb{Z} = \phi, \\ a_n, & \text{if } (x_{i-1}, x_i] \cap \mathbb{Z} = \{n\} \end{cases}$$

Once $m(\underline{x}) < 1$. So

$$\sum_{i=1}^{N} f(\xi_i)\left(A(x_i) - A(x_{i-1})\right) = \text{lots of } 0\text{s}$$

$$+ \sum_{c < n \le d} f(\xi_i)\, a_n$$

for some ξ_i close to n.

Use continuity ...

$\boxed{\text{II}}$ Integration by parts (Theorem A.2):

$$\int_c^d f(x)\,dg(x) = f(x)g(x)\Big|_c^d - \int_c^d g(x)\,df(x)$$

$$= \big(f(d)g(d) - f(c)g(c)\big) - \int_c^d g(x)\,df(x).$$

Comes from the identity (summation by parts): set $x_0 = c$, $x_{N+1} = d$.

$$\sum_{n=1}^{N} g(\xi_n)\big(f(x_n) - f(x_{n-1})\big)$$

$$= f(x_N)g(d) - f(c)g(c)$$

$$- \sum_{n=1}^{N+1} f(x_{n-1})\big(g(\xi_n) - g(\xi_{n-1})\big).$$

partition is ξ, sample points x_n

$\boxed{\text{III}}$ "Un-Stieltjesification" (Thm A.3):

If g is Riemann integrable, and f is continuously differentiable, then

$$\underbrace{\int_c^d g(x)\,df(x)}_{\text{R-S}} = \underbrace{\int_c^d g(x) f'(x)\,dx}_{\text{Riemann}}.$$

Mean Value Theorem:

$$f\cancel{g}(x_i) - f\cancel{g}(x_{i-1}) = f\cancel{g}'(u_i)(x_i - x_{i-1})$$

for some $u_i \in [x_{i-1}, x_i]$.

Dirichlet series: $\sum_{n=1}^{\infty} a_n n^{-s}$, $s \in \mathbb{C}$

Compare to power series (centred at $z=0$): $\sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{C})$.

- converges in some disk of radius R (possibly $R=0$ or $R=\infty$)
- for $|z|<R$, converges absolutely
- for $|z|<R$, converges "locally uniformly" (for example, uniformly for $\{|z| \le r\}$ if $r<R$.)
  - ↳ implies that we can differentiate term-by-term (power series are analytic)
- can be analytically continued beyond $\{|z|<R\}$.

- $\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty} c_n z^n$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ ("additive convolution"). $\to = \sum_{j+k=n} a_j b_k$

Dirichlet series: We will show

- converges in a right half-plane $\{\operatorname{Re} \cancel{z}\, s > R\}$ (possibly $R=-\infty$ or $R=\infty$)
- sometimes converge conditionally

  Example: $\sum_{n=1}^{\infty} (-1)^n n^{-1/2}$.
- converge locally uniformly $\Rightarrow$ analytic
- $\left(\sum_{n=1}^{\infty} a_n n^{-s}\right)\left(\sum_{n=1}^{\infty} b_n n^{-s}\right) = \sum_{n=1}^{\infty} c_n n^{-s}$

where $c_n = \sum_{d \mid n} a_d b_{n/d}$ ("multiplicative convolution") ↳ $\sum_{j \cdot k = n} a_j b_k$

assuming absolute convergence

Notation: We write complex variables as

$$s = \sigma + it, \quad \sigma, t \in \mathbb{R}; \text{ so}$$

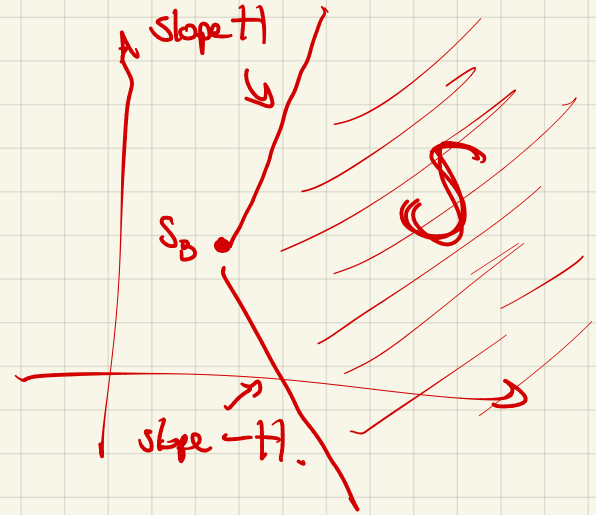
$$\sigma = \mathrm{Re}(s) \text{ and } t = \mathrm{Im}(s). \qquad (s_0 = \sigma_0 + it_0)$$

- example: first quadrant is $\{s : \sigma > 0,\ t > 0\}$
- example: if $x > 0$, then

$$|x^s| = |x^{\sigma+it}| = |x^\sigma|\,|\underbrace{x^{it}}_{e^{it\log x}}| = x^\sigma \cdot 1$$

Theorem 1.1: Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series $(a_n \in \mathbb{C})$. Suppose that the series converges at $s = s_0$. Then, for any $H > 0$, the series converges uniformly in the sector

$$S = \{s \in \mathbb{C} : \sigma \ge \sigma_0,\ |t - t_0| \le H(\sigma - \sigma_0)\}.$$



Consequences:

- If $\alpha(s_0)$ converges, then $\alpha(s)$ converges for all s with $\sigma > \sigma_0$.
- If $\alpha(s_0)$ diverges, then $\alpha(s)$ diverges for all s with $\sigma < \sigma_0$.
- $\alpha(s)$ has an abscissa of convergence: a real number σ_c such that $\alpha(s)$ converges when $\sigma > \sigma_c$ and diverges when $\sigma < \sigma_c$. (possibly $\sigma_c = \pm\infty$) (on $\sigma = \sigma_c$, unpredictable)
- $\alpha(s)$ converges locally uniformly on $\{\sigma > \sigma_c\}$, hence is always analytic there.

· we can differentiate term by term:

$$\alpha'(s) = \sum_{n=1}^{\infty} \frac{d}{ds}(a_n n^{-s}) = \sum_{n=1}^{\infty} a_n n^{-s}(-\log n).$$

(σ_c is the same for $\alpha(s)$ and $\alpha'(s)$)

Example: the Riemann zeta function.

$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. We know $\zeta(x)$ converges when $x>1$ and diverges when $x\le 1$. Thus $\sigma_c = 1$: $\zeta(s)$ converges for all $\sigma>1$ and diverges for all $\sigma<1$.

Also, $\zeta'(s) = -\sum_{n=1}^{\infty}(\log n)n^{-s}$.

Write $A(x) = \alpha(0) - \sum_{n>x} a_n = \alpha(0) - R(x)$,

so that $R(x) = o(1)$.

Hence $R(x) \ll 1$.

Theorem 1.1: $(a_n \in \mathbb{C})$

Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series. Suppose that the series converges at $s=s_0$. Then, for any $H>0$, the series converges uniformly in the sector

$$S = \{ s\in\mathbb{C} : \sigma \ge \sigma_0, \ |t-t_0| \le H(\sigma-\sigma_0) \}.$$

Proof: By the change of variables

$s \rightsquigarrow s+s_0$ (which changes a_n to $a_n n^{-s_0}$), we can assume $s_0=0$.

Write $A(x) = \sum_{n\le x} a_n$, so that

$$\lim_{x\to\infty} A(x) = \sum_{n=1}^{\infty} a_n n^{-0} = \alpha(0) \text{ converges.}$$

For $\sigma > 0$,

$$\sum_{M<n\le N} a_n n^{-s} = \int_M^N t^{-s}\,dA(t) \qquad \boxed{I}$$

$$= \int_M^N t^{-s}\,d(\alpha(0) - R(t))$$

$$= \cancelto{0}{\int_M^N t^{-s}\,d\alpha(0)} - \int_M^N t^{-s}\,dR(t).$$

by $\boxed{III}$

Integration by parts: $\boxed{II}$

$$\sum_{M<n\le N} a_n n^{-s} = -t^{-s}R(t)\Big|_M^N + \int_M^N R(t)\,d(t^{-s}) \qquad \boxed{III}$$

$$= R(M)M^{-s} - R(N)N^{-s} + \int_M^N R(t)(-st^{-s-1}\,dt).$$

As $N\to\infty$, $R(N)N^{-s} \lesssim 1\cdot N^{-\sigma}$ goes to 0; and

$$\int_N^\infty R(t)(-st^{-s-1})\,dt \lesssim_s \int_N^\infty 1\cdot t^{-\sigma-1}\,dt$$

$$= \frac{N^{-\sigma}}{\sigma} \to 0 \text{ as } N\to\infty.$$

Hence $\displaystyle\sum_{n>M} a_n n^{-s} = R(M)M^{-s} - s\int_M^\infty R(t)t^{-s-1}\,dt.$

Choose M large enough that $|R(t)| < \varepsilon$ for all $t \ge M$. Then

$$\left|\sum_{n>M} a_n n^{-s}\right| \le \frac{\varepsilon}{M^\sigma} + |s|\int_M^\infty \varepsilon\, t^{-\sigma-1}\,dt$$

$$= \frac{\varepsilon}{M^\sigma} + \frac{|s|}{\sigma}\varepsilon M^{-\sigma}.$$

Exercise: for $s \in S = \{s : \sigma \ge 0,\ |t| \le H\sigma\}$,

$$|s|/\sigma \le \sqrt{1+H^2},$$

So $$\left|\sum_{n>M}\right| \le \frac{\varepsilon}{M^\sigma}\left(1+\sqrt{1+H^2}\right)$$

uniformly for $s \in S$. //

We want to relate $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, $A(x) = \sum_{n \le x} a_n$.

<u>Theorem</u>: Suppose $\alpha(s)$ has abscissa of convergence σ_c with $\sigma_c \ge 0$. Then for $\sigma > \sigma_c$, we have

$$\alpha(s) = s\int_1^{\infty} A(x) x^{-s-1}\,dx. \quad (*)$$

Moreover,

$$\limsup_{x\to\infty} \frac{\log|A(x)|}{\log x} = \sigma_c.$$

<u>Notation issue</u>: Note that

$$\int_1^N x^{-s}\,dA(x) = \sum_{1<n\le N} a_n n^{-s} = \sum_{n=2}^{N} a_n n^{-s},$$

while for $\delta \in (0,1)$, $\int_\delta^N x^{-s}\,dA(x) = \sum_{n=1}^{N} a_n n^{-s}$.

We'll write $\int_{1^-}^N x^{-s}\,dA(x) = \lim_{\delta\to 1^-} \int_\delta^N x^{-s}\,dA(x)$.

Proof of $(*)$:

$$\sum_{n=1}^{N} a_n n^{-s} = \int_{1^-}^{N} x^{-s}\,dA(x) \quad \boxed{\mathrm{I}}$$

$$= x^{-s}A(x)\Big]_{1^-}^{N} - \int_{1^-}^{N} A(x)\,d(x^{-s}) \quad \boxed{\mathrm{II}}$$

$$= \frac{A(N)}{N^s} - 0 - \int_{1^-}^{N} A(x)(-s x^{-s-1}\,dx) \quad \boxed{\mathrm{III}}$$

$$= \frac{A(N)}{N^s} + s\int_{\cancel{1^-}\ 1}^{N} A(x)\,x^{-s-1}\,dx,$$

since $A(x)=0$ for $x<1$.

~~Take limits as $N\to\infty$. (LHS ok since $\sigma > \sigma_c$.) $\sum_{n=1}^{\infty} a_n n^{-s}$~~

We'll proceed from here on Thursday.