Thursday, January 16

- Suggested Problems #1 posted today
- Group Work #2 in class on Tuesday

FROM TUESDAY:

Theorem 1.1: $(a_n \in \mathbb{C})$

Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series.

Suppose that the series converges at $s = s_0$.

Then, for any $H > 0$, the series converges uniformly in the sector

$$S = \{ s \in \mathbb{C} : \sigma \geq \sigma_0,\ |t - t_0| \leq H(\sigma - \sigma_0) \}.$$

Consequence:

- $\alpha(s)$ has an abscissa of convergence: a real number σ_c such that $\alpha(s)$ converges when $\sigma > \sigma_c$ and diverges when $\sigma < \sigma_c$.

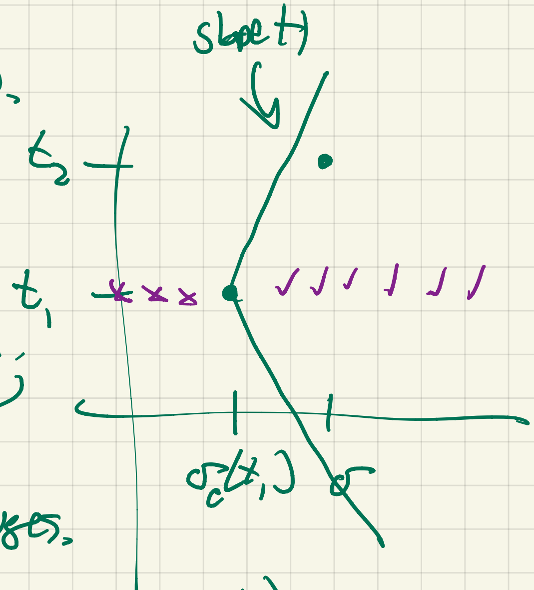
In fact,

$$\sigma_c = \inf \{ \sigma \in \mathbb{R} : \alpha(\sigma) \text{ converges} \}.$$

Proof: Let's define

$$\sigma_c(t) = \inf \{ \sigma \in \mathbb{R} : \alpha(\sigma + it) \text{ converges} \}$$

We claim $\sigma_c(t)$ is constant.



Take $\sigma > \sigma_c(t_1)$.

There's some $H > 0$ such that $\sigma + it_2$ is in the H-sector; thus by Th^m 1.1, $\alpha(\sigma + it_2)$ converges.

Consequently, $\sigma_c(t_2) \leq \sigma_c(t_1)$.

The exact same argument shows $\sigma_c(t_1) \leq \sigma_c(t_2)$.

$A(x) = \sum_{n\le x} a_n$ $\qquad$ $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$

Theorem 1.3: Suppose $\alpha(s)$ has abscissa of convergence σ_c with $\sigma_c \ge 0$. Then for $\sigma > \sigma_c$, we have

$$\alpha(s) = s\int_1^{\infty} A(x)x^{-s-1}\,dx. \quad (*)$$

Moreover,

$$\limsup_{x\to\infty} \frac{\log|A(x)|}{\log x} = \sigma_c.$$

WE SHOWED ON TUESDAY THAT

$$\sum_{n=1}^{N} a_n n^{-s} = \frac{A(N)}{N^s} + s\int_1^N A(x)x^{-s-1}\,dx. \quad (**)$$

Define $\phi = \limsup_{x\to\infty} \frac{\log|A(x)|}{\log x}$.

- Suppose $\sigma = \phi + \varepsilon$ for some $\varepsilon > 0$.

Then $\frac{\log|A(x)|}{\log x} < \phi + \frac{\varepsilon}{2}$ for x large enough,

or $|A(x)| < x^{\phi+\varepsilon/2}$ for large x.

Then $\frac{A(N)}{N^\sigma} \ll \frac{N^{\phi+\varepsilon/2}}{N^\sigma} = N^{-\varepsilon/2}$; and

$$\int_N^\infty A(x)x^{-\sigma-1}\,dx \ll \int_N^\infty x^{\phi+\varepsilon/2}\,x^{-(\phi+\varepsilon)-1}\,dx$$

$$= \int_N^\infty x^{-1-\varepsilon/2}\,dx = \frac{N^{-\varepsilon/2}}{\varepsilon/2} \ll_\varepsilon N^{-\varepsilon/2}.$$

Thus from (**),

$$\sum_{n=1}^{N} a_n n^{-\sigma} = O(N^{-\varepsilon/2}) + s\left(\int_1^\infty A(x)x^{-\sigma-1}\,dx + O_\varepsilon(N^{-\varepsilon/2})\right).$$

Take $N\to\infty$:

$$\alpha(\sigma) = \sum_{n=1}^{\infty} a_n n^{-\sigma} = \int_1^\infty A(x)x^{-\sigma-1}\,dx,$$

In particular, $\alpha(\sigma)$ converges

$\Rightarrow \sigma_c \le \sigma = \phi + \varepsilon$. Since this holds for all $\varepsilon > 0$, we get $\sigma_c \le \phi$.

• Conversely, let $\sigma_0 = \sigma_c + \varepsilon$ for some $\varepsilon > 0$. We have the identity

$$A(N) = -R_0(N)N^{\sigma_0} + \sigma_0 \int_0^N R_0(x)\, x^{\sigma_0 - 1}\, dx$$

(from Theorem 1.1 proof, setting $M=0$), where $R_0(N) = \sum_{n>N} a_n n^{-\sigma_0}$ is $o(1)$ and thus $\ll 1$. Hence

$$A(N) \ll_{\sigma_0} 1 \cdot N^{\sigma_0} + \int_0^N 1 \cdot x^{\sigma_0 - 1}\, dx$$

$$\ll_{\sigma_0} N^{\sigma_0},$$

which implies $\limsup_{x \to \infty} \frac{\log |A(x)|}{\log x} \le \sigma_0 = \sigma_c + \varepsilon$.

So $\phi \le \sigma_c + \varepsilon$ for every $\varepsilon > 0$, and so $\phi \le \sigma_c$.

Here's a Dirichlet series:

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} = 1 - 2^{-s} + 3^{-s} - 4^{-s} + \cdots$$

• Where does $\eta(s)$ converge absolutely?

$\sigma > 1$: $\sum_{n=1}^{\infty} \left| (-1)^{n-1} n^{-s} \right| = \sum_{n=1}^{\infty} n^{-\sigma}$ ✓

(b) For real σ, where does $\eta(\sigma)$ converge?

$\sigma > 0$: Alternating series test (+ Test for Divergence)

• For complex s, where does $\eta(s)$ converge?

$\sigma > 0$: because $\sigma_c(a) = 0$ by (b), the half-plane of convergence is $\{\sigma > 0\}$.

Weird outcome: $\eta(s)$ converges conditionally for $\{0 < \sigma \le 1\}$.

Definition: Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. The abscissa of absolute convergence, σ_a, of $\alpha(s)$ is defined to be the abscissa of convergence of $\sum_{n=1}^{\infty} |a_n| n^{-s}$.

$$\sigma_a = \inf \left\{ \sigma : \sum_{n=1}^{\infty} |a_n| n^{-\sigma} \text{ converges} \right\}$$

Example: For $\eta(s)$, $\sigma_c = 0$ and $\sigma_a = 1$.

Example: If $a_n \geq 0$ for all n, then $\sigma_c = \sigma_a$.

Theorem 1.4: Always $\sigma_c \leq \sigma_a \leq \sigma_c + 1$.

Proof: Since absolute convergence implies convergence, $\sigma_c \leq \sigma_a$ is trivial.

Let $\sigma = \sigma_c + 1 + \varepsilon$; we want to show that $\alpha(\sigma)$ converges absolutely.

Note that $\alpha(\sigma_c + \varepsilon/2) = \sum_{n=1}^{\infty} a_n n^{-\sigma_c - \varepsilon/2}$ converges (by definition of σ_c); by the Test for Divergence, $\lim_{n\to\infty} a_n n^{-\sigma_c - \varepsilon/2} = 0$.

Hence

$$\sum_{n=1}^{\infty} |a_n| n^{-\sigma} = \sum_{n=1}^{\infty} |a_n n^{-\sigma_c - \varepsilon/2}| n^{-1-\varepsilon/2}$$

$$\ll \sum_{n=1}^{\infty} 1 \cdot n^{-1-\varepsilon/2} \text{ converges,}$$

So $\sigma_a \leq \sigma_c + 1 + \varepsilon$ for every $\varepsilon > 0$, which is enough.

Remark: every ordered pair (σ_c, σ_a) satisfying $\sigma_c \leq \sigma_a \leq \sigma_c + 1$ is possible.

· Is σ_c or σ_a determined by the "rightmost singularity" of $\alpha(s)$?

No (maybe surprisingly):

· $\eta(s)$ is analytic for $\sigma > 0$, so no singularities on $\sigma = 1 = \sigma_a$.

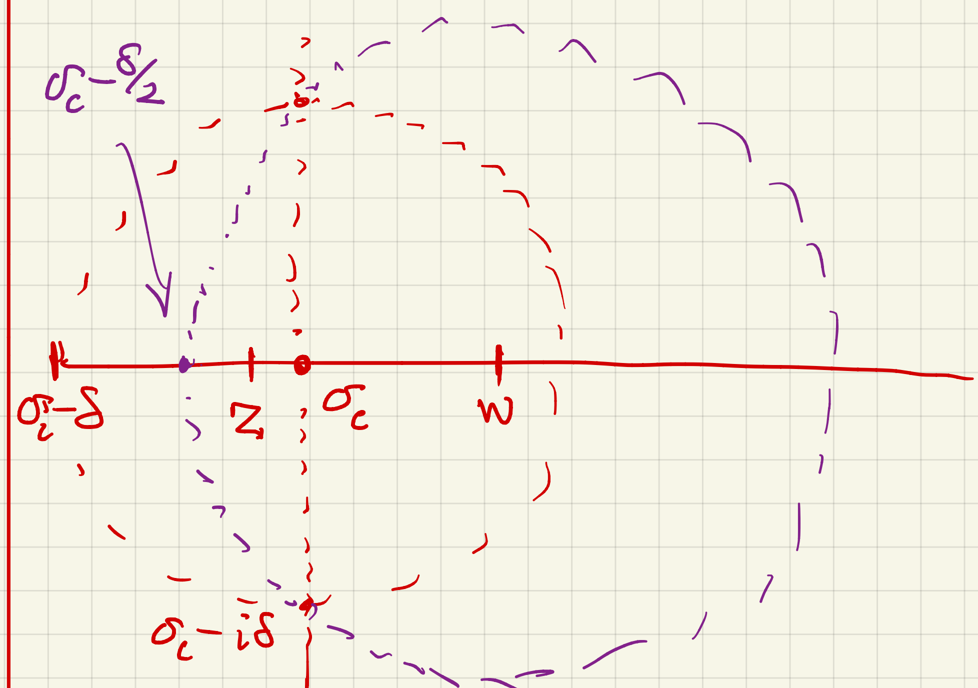
· We'll prove later that $\eta(s)$ has an analytic continuation to an <u>entire</u> function!

One nice exception:

<u>Theorem 1.7</u> (Landau's theorem): Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ with $a_n \geq 0$ for all (but finitely many) $n \in \mathbb{N}$. Then $\alpha(s)$ has a singularity at $\sigma = \sigma_c$.

("singularity" means "can't be analytically continued there")

<u>Proof</u>: Suppose not, so $\alpha(s)$ has an analytic continuation to $\{s \in \mathbb{C} : |s - \sigma_c| < \delta\}$. Define $z = \sigma_c - \delta/4$, $w = \sigma_c + 3\delta/4$. Then $\alpha(s)$ is analytic on $\{s \in \mathbb{C} : |s - w| < \frac{5\delta}{4}\}$.



We have the power series

$$\alpha(z) = \sum_{k=0}^{\infty} \frac{\alpha^{(k)}(w)}{k!} (z-w)^k$$

Since w is in the half-plane of convergence of $\alpha(s)$, we can differentiate term by term k times:

$$\alpha^{(k)}(w) = \sum_{n=1}^{\infty} a_n n^{-w} (-\log n)^k .$$

Hence

$$\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{k!} (w-z)^k \sum_{n=1}^{\infty} a_n (\log n)^k n^{-w}$$

Everything in sight is nonnegative! so we can interchange summations

$$\alpha(z) = \sum_{n=1}^{\infty} a_n n^{-w} \sum_{k=0}^{\infty} \frac{1}{k!} ((w-z)\log n)^k$$

$$= \sum_{n=1}^{\infty} a_n n^{-w} \cdot e^{(w-z)\log n}$$

$$= \sum_{n=1}^{\infty} a_n n^{-w} \, n^{w-z} = \sum_{n=1}^{\infty} a_n n^{-z} .$$

In particular, the series

$$\sum_{n=1}^{\infty} a_n n^{-s} \text{ converges at } s=z;$$

but $z < \sigma_c$, a contradiction.

Motivating examples:

(1) Recall $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Is this legal?

$$\zeta(s)^2 = \left(\sum_{m=1}^{\infty} m^{-s}\right)\left(\sum_{n=1}^{\infty} n^{-s}\right)$$

$$= \sum_{m,n\in\mathbb{N}} m^{-s} n^{-s} = \sum_{k=1}^{\infty} k^{-s} \#\left\{\begin{array}{c}(m,n)\in\mathbb{N}^2: \\ mn=k\end{array}\right\}$$

$$= \sum_{k=1}^{\infty} d(k) k^{-s}.$$

(2) Note that

$$(1+2^{-s}+4^{-s}+\cdots+2^{-100s})(1+3^{-s}+\cdots+3^{-100s}) \times$$

$$\times(1+5^{-s}+\cdots+5^{-100s}) = \sum_{n\in W} n^{-s}$$

where $W = \{2^a 3^b 5^c : 0 \le a,b,c \le 100\}$.

Is it legal to do the doubly-infinite version of this:

$$\prod_{\text{primes } p} (1+p^{-s}+p^{-2s}+\cdots)$$

$$= \sum_{\text{all } n} n^{-s} = \zeta(s)?$$

Both yes — assuming absolute convergence

Notation: given (a_n) and (b_n), define (c_n) by $c_n = \sum_{de=n} a_d b_e = \sum_{d \mid n} a_d b_{n/d}$.

This is the Dirichlet convolution:

$$c = a * b.$$

Theorem 1.8 (exercise):

Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and

$\beta(s) = \sum_{n=1}^{\infty} b_n n^{-s}$. Define $c = a * b$

and set $\gamma(s) = \sum_{n=1}^{\infty} c_n n^{-s}$.

If $\alpha(s)$ and $\beta(s)$ both converge

absolutely, then so does $\gamma(s)$

and $\gamma(s) = \alpha(s)\beta(s)$. //