Tuesday, January 21

Group Work #2 today

Recall:

Theorem 1.8:

Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and

$\beta(s) = \sum_{n=1}^{\infty} b_n n^{-s}$. Define $c = a * b$ $\quad c_n = \sum_{d \mid n} a_d b_{n/d}$

and set $\gamma(s) = \sum_{n=1}^{\infty} c_n n^{-s}$.

If $\alpha(s)$ and $\beta(s)$ both converge absolutely, then so does $\gamma(s)$ and $\gamma(s) = \alpha(s)\beta(s)$.

<u>Example</u>: Let $M(s) = \sum_{n=1}^{\infty} \mu(n) n^{-s}$, where μ is the Möbius function. What is $M(s)\zeta(s)$?

- Note that $\left| \sum_{n=1}^{\infty} \mu(n) n^{-s} \right| \leq \sum_{n=1}^{\infty} |\mu(n)| n^{-\sigma} \leq \sum_{n=1}^{\infty} 1 \cdot n^{-\sigma}$; so both $M(s)$ and $\zeta(s)$ converge absolutely for $\sigma > 1$.

Thus for $\sigma > 1$,

$$M(s)\zeta(s) = \sum_{n=1}^{\infty} (\mu * 1)(n) n^{-s}.$$

But $(\mu * 1)(n) = \sum_{d \mid n} \mu(d) \cancel{1(n/d)} = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise} \end{cases}$

So $M(s)\zeta(s) = 1 \cdot 1^{-s} + \sum_{n=2}^{\infty} 0 \cdot n^{-s} = 1.$

In other words, $M(s) = 1/\zeta(s)$ for $\sigma > 1$.

<u>Consequence</u>: $\zeta(s) \neq 0$ on $\{\sigma > 1\}$.

Indeed, if $F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$ and $G(s) = \sum_{n=1}^{\infty} g(n) n^{-s}$, then (where things converge absolutely)

$F(s)\zeta(s) = G(s)$ if and only if $F(s) = G(s)/\zeta(s) = G(s)M(s)$.

$\Updownarrow$

$g = f * 1$ if and only if $f = g * \mu$.

— Möbius inversion formula.

Example: The identity $\sum_{d|n} \phi(d) = n$ is $\phi * 1 = $ (identity function). Hence

$$\left(\sum_{n=1}^{\infty} \phi(n) n^{-s}\right)\left(\sum_{n=1}^{\infty} 1 \cdot n^{-s}\right) = \left(\sum_{n=1}^{\infty} n \cdot n^{-s}\right)$$

or

$$\sum_{n=1}^{\infty} \phi(n) n^{-s} = \zeta(s-1)/\zeta(s).$$

Exercise: Show the above calculation holds for $\sigma > 2$, as does

$$\sum_{n=1}^{\infty} \sigma(n) n^{-s} = \zeta(s-1)\zeta(s),$$

where $\sigma(n)$ is the sum-of-divisors function.

Recall: $f(n)$ is multiplicative if $f(mn) = f(m) f(n)$ whenever $(m,n) = 1$.

Theorem 1.9: Let $F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$ where f is multiplicative. Then

$$F(s) = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \frac{f(p^3)}{p^{3s}} + \cdots\right)$$

whenever $\sigma > \sigma_a$ (that is, whenever $\sum_{n=1}^{\infty} |f(n)| n^{-\sigma}$ converges).

Euler products

Example: $\mu(n)$ is multiplicative. Then for $\sigma > 1$,

$$M(s) = \sum_{n=1}^{\infty} \mu(n) n^{-s}$$

$$= \prod_p \left(1 + \frac{\mu(p)}{p^s} + \frac{\mu(p^2)}{p^{2s}} + \frac{\mu(p^3)}{p^{3s}} + \cdots\right)$$

$$= \prod_p \left(1 + \frac{-1}{p^s} + 0 + 0 + \cdots\right) = \prod_p (1 - p^{-s}).$$

Consequently,

$$\zeta(s) = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots\right)$$

$$= \prod_p (1 - p^{-s})^{-1}.$$

More generally:

Exercise: Show that if f is totally multiplicative, then

$$\sum_{n=1}^{\infty} f(n) n^{-s} = \prod_p (1 - f(p) p^{-s})^{-1}$$

in the half plane of absolute convergence.