Thursday, January 23

RECALL:

Theorem 1.9: Let $F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$ where f is multiplicative. Then $F(s)$ has the Euler product

$$F(s) = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \frac{f(p^3)}{p^{3s}} + \cdots\right)$$

whenever $\sigma > \sigma_a$ (that is, whenever $\sum_{n=1}^{\infty} |f(n)| n^{-\sigma}$ converges). ∎

Remark: Almost the same proof shows that these Euler products converge absolutely; in particular, the product can't equal 0 unless one of the factors equals 0.

Detour: the definition of a product converging absolutely is:

"$\prod (1 + a_n)$ converges absolutely" means $\sum |a_n|$ converges.

Heuristic for Theorem 1.9:

$$\prod = \left(1 + f(2) 2^{-s} + f(2^2) 2^{-2s} + \cdots\right)$$
$$\left(1 + f(3) 3^{-s} + f(3^2) 3^{-2s} + \cdots\right)$$
$$\vdots$$

- $1 \cdot 1 \cdot 1 \cdot f(7^2) 7^{-2s} \cdot f(11) 11^{-s} \cdot 1 \cdot 1 \cdot 1 \cdots$
  $= f(539) 539^{-s}$.
- $1 \cdot f(3) 3^{-s} \cdot 1 \cdot f(7) 7^{-s} \cdot 1 \cdot f(13) 13^{-s} \cdots$
- $f(2^2) 2^{-2s} f(3^3) 3^{-3s} f(5^5) 5^{-5s} \cdots$

$$\sum_{n=1}^{\infty} f(n) n^{-s} = \prod_{p} \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \frac{f(p^3)}{p^{3s}} + \cdots\right)$$

Proof: For each prime p,

$$\left|1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\right| \le \sum_{k=0}^{\infty} |f(p^k) p^{-ks}|$$

$$\le \sum_{n=1}^{\infty} |f(n)| n^{-\sigma} < \infty$$

by assumption. So we can "infinite FOIL" a finite number of factors:

$$\prod_{p \le y} \left(1 + \frac{f(p)}{p^s} + \cdots\right) = \sum_{n \text{ } y\text{-friable}} f(n) n^{-s},$$

where the y-friable numbers (also called y-smooth) are $\{n \in \mathbb{N} : p \mid n \Rightarrow p \le y\}$.

Then

$$\left| F(s) - \prod_{p \le y} \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\right) \right|$$

$$= \left| \sum_{n=1}^{\infty} f(n) n^{-s} - \sum_{n \text{ } y\text{-friable}} f(n) n^{-s} \right|$$

$$= \left| \sum_{\substack{n \text{ not} \\ y\text{-friable}}} f(n) n^{-s} \right| \le \sum_{\substack{n \text{ not} \\ y\text{-friable}}} |f(n)| n^{-\sigma}$$

$$< \sum_{n > y} |f(n)| n^{-\sigma},$$

which tends to 0 as $y \to \infty$ since $F(s)$ is absolutely convergent.

Hence $\lim_{y \to \infty} \prod_{p \le y} \left(1 + \frac{f(p)}{p^s} + \cdots\right) = F(s)$. //

Notation:

- $\omega(n) = \#\{\text{distinct prime factors of } n\}$
- $\Omega(n) = \#\{\text{prime factors of } n \text{ with multiplicity}\}$

Example: $539 = 7^2 \cdot 11$, so $\omega(539) = 2$ and $\Omega(539) = 3$.

- The Liouville lambda function is

$\lambda(n) = (-1)^{\Omega(n)}$. (Note that $\lambda(n) = \mu(n)$ if and only if n is squarefree.)

Let's find an Euler product for $\sum_{n=1}^{\infty} \lambda(n) n^{-s}$ (since $\lambda(n)$ is totally multiplicative).

Note that

$$\sum_{n=1}^{\infty} |\lambda(n) n^{-s}| = \sum_{n=1}^{\infty} n^{-\sigma}, \text{ so } \sigma_a = 1$$

So for $\sigma > 1$, by the theorem,

$$\sum_{n=1}^{\infty} \lambda(n) n^{-s} = \prod_p \left(1 + \frac{\lambda(p)}{p^s} + \frac{\lambda(p^2)}{p^{2s}} + \cdots\right)$$

$$= \prod_p \left(1 - \frac{1}{p^s} + \frac{1}{p^{2s}} - \frac{1}{p^{3s}} + \frac{1}{p^{4s}} \cdots\right)$$

$$= \prod_p \frac{1}{1 - (-1/p^s)} = \prod_p (1 + p^{-s})^{-1}.$$

Exercise: Show that $\sum_{n=1}^{\infty} \lambda(n) n^{-s} = \frac{\zeta(2s)}{\zeta(s)}$ for $\sigma > 1$.

Example: For $\sigma > 1$, the Euler product

$$\zeta(s) = \prod_p (1-p^{-s})^{-1}$$ converges absolutely,

and so

$$\log \zeta(s) = \sum_p \log(1-p^{-s})^{-1}$$

$$= \sum_p \sum_{k=1}^{\infty} \frac{(p^{-s})^k}{k}$$

$$= \sum_{p^k} \frac{1}{k} \frac{1}{(p^k)^{-s}}$$

(by absolute convergence). This is a Dirichlet series: $\log \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \begin{cases} 1/k, & \text{if } n=p^k, \\ 0, & \text{otherwise.} \end{cases}$

Take derivatives of both sides

(term-by-term on the RHS, by

local uniform convergence of Dirichlet series)

$$\frac{d}{ds}(\log \zeta(s)) = \sum_{n=1}^{\infty} \frac{d}{ds} n^{-s} \begin{cases} 1/k, & \text{if } n=p^k, \\ 0, & \text{otherwise} \end{cases}$$

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} (-\log n)\, n^{-s} \begin{cases} 1/k, & \text{if } n=p^k, \\ 0, & \text{otherwise.} \end{cases}$$

Define $\Lambda(n)$ to be the von Mangoldt Lambda function

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n=p^k \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}.$$

Exercise: Show that $\sum_{d|n} \Lambda(d) = \log n.$

Thus $\Lambda * 1 = \log$; so for $\sigma > 1$,

$$-\frac{\zeta'}{\zeta}(s) \cdot \zeta(s) = \sum_{n=1}^{\infty} \log n \cdot n^{-s} = -\zeta'(s).$$

Recall $\eta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}$.

We'll prove twice that $\eta(s) = (1-2^{1-s})\zeta(s)$ for $\sigma > 1$.

(1) For $\sigma > \sigma_a = 1$,

$$\eta(s) = 1 + (2^{-s} - 2\cdot 2^{-s}) + 3^{-s} + (4^{-s} - 2\cdot 4^{-s}) + 5^{-s} + (6^{-s} - 2\cdot 6^{-s}) \cdots$$

$$= (1 + 2^{-s} + 3^{-s} + 4^{-s} + \cdots) - 2(2^{-s} + 4^{-s} + 6^{-s} + \cdots)$$

$$= \zeta(s) - 2\cdot 2^{-s}(1^{-s} + 2^{-s} + 3^{-s} + \cdots)$$

$$= \zeta(s) - 2\cdot 2^{-s}\cdot \zeta(s).$$

✓

(2) Note that $(-1)^{n-1}$ is multiplicative!

- it takes the value -1 on any 2^k and the value $+1$ on any odd p^k.

So for $\sigma > 1$ we have the Euler product

$$\eta(s) = \prod_p \left(1 + \frac{\eta(p)}{p^s} + \frac{\eta(p^2)}{p^{2s}} + \cdots\right)$$

$$= \left(1 - \frac{1}{2^s} - \frac{1}{2^{2s}} - \cdots\right) \prod_{p \geq 3} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right)$$

$$= \frac{\left(1 - \frac{1}{2^s} - \frac{1}{2^{2s}} - \cdots\right)}{\left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \cdots\right)} \prod_{p \geq 2} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right)$$

$$= \frac{1 - \frac{1}{2^s}\left(1 - 1/2^s\right)^{-1}}{\left(1 - 1/2^s\right)^{-1}} \zeta(s)$$

✓.

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} = (1-2^{1-s})\zeta(s).$$

Observation: We proved this identity for $\sigma > 1$. But for $\eta(s)$ we saw that $\sigma_c = 0$, so that $\eta(s)$ is analytic for $\sigma > 0$. Thus

(*) $\zeta(s) = \eta(s)/(1-2^{1-s})$ is ~~an analytic~~ a meromorphic continuation of $\zeta(s)$ to $\sigma > 0$

denominator has zeros at $s = 1 + \frac{2\pi i}{\log 2} k$, all $k \in \mathbb{Z}$.

Exercise: show using (*) that $\zeta(s)$ has a simple pole at $s=1$ with residue 1.

Notation: $\{x\} = x - \lfloor x \rfloor$ is the fractional-part function. For example, $\{1.23\} = 0.23$, $\{-1.23\} = 0.77$, $\{\pi\} = \pi - 3$, $\{539\} = 0$.

Flexible representation of $\zeta(s)$:

Theorem 1.12: For any $x > 0$, and for any $s \neq 1$ with $\sigma > 0$,

$$\zeta(s) = \sum_{n \le x} n^{-s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s\int_x^{\infty} \{u\} u^{-s-1} du. \quad (**)$$

Proof: Start by assuming $\sigma > 1$. Then

$$\zeta(s) = \sum_{n \le x} n^{-s} + \sum_{n > x} n^{-s}.$$

Now $\sum_{n>x} n^{-s} = \int_x^\infty u^{-s}\, d\lfloor u\rfloor$

$$= \int_x^\infty u^{-s}\, d(u) - \int_x^\infty u^{-s}\, d\{u\}$$

$$= \frac{u^{1-s}}{1-s}\Big|_x^\infty - \left(u^{-s}\{u\}\Big|_x^\infty - \int_x^\infty \{u\}\, d(u^{-s}) \right)$$

$$= 0 - \frac{x^{1-s}}{1-s} - \left(0 - \{x\}x^{-s}\right)$$

$$+ \int_x^\infty \{u\}(-s)u^{-s-1}\, du.$$

This proves (**) for $\sigma > 1$. But

$\sum_{n\le x} n^{-s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s}$ is analytic everywhere except $s=1$; moreover,

$$\left| \int_x^\infty \{u\} u^{-s-1}\, du \right| \le \int_x^\infty 1\, u^{-\sigma-1}\, du$$

converges for $\sigma > 0$; hence the RHS of (**) is analytic everywhere in $\{\sigma > 0\}$ except $s = 1$.

So (**) holds by analytic continuation.

Consequences of (**):

- When $x = 1$, we get

$$\zeta(s) = 1 + \frac{1}{s-1} + 0 - s\int_1^\infty \{u\} u^{-s-1}\, du.$$

Thus $\zeta(s) - \frac{1}{s-1}$ has a removable singularity at $s=1$, with value

$$C_0 = 1 - \int_1^\infty \{u\} u^{-2}\, du \approx 0.572.$$

Thus $\zeta(s) = \frac{1}{s-1} + C_0 + O(|s-1|)$ near $s = 1$.

• Also for $\sigma > 0$, $s \neq 1$:

$$\sum_{n \le x} n^{-s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O\left(\frac{1}{x^{\sigma}} + |s| \int_x^\infty 1 \cdot u^{-\sigma-1}\,du\right).$$

$$= \frac{x^{1-s}}{1-s} + \zeta(s) + O\left(x^{-\sigma}\left(1 + \frac{|s|}{\sigma}\right)\right).$$

So: if $s > 1$, then

$$\sum_{n \le x} n^{-s} = \zeta(s) + O_s\left(x^{1-s}\right).$$

if $0 < s < 1$, then

$$\sum_{n \le x} n^{-s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O_s\left(x^{-s}\right).$$

Example:

$$\sum_{n \le x} \frac{1}{\sqrt{n}} = 2\sqrt{x} + \zeta\left(\tfrac{1}{2}\right) + O\left(\frac{1}{\sqrt{x}}\right).$$