Tuesday, January 28

Group Work #3 in class on Thursday

RECALL:

$$\boxed{\zeta(s) = \frac{1}{s-1} + 1 - s\int_1^\infty \{u\} u^{-s-1}\,du.}$$

Corollary 1.14: For $0 < s < 1$,

$$\frac{1}{s-1} < \zeta(s) < \frac{s}{s-1} < 0.$$

Proof: Since $0 \le \{u\} < 1$, we have

$$0 \le s\int_1^\infty \{u\} u^{-s-1}\,du \le s\int_1^\infty 1 \cdot u^{-s-1}\,du$$

$$= -u^{-s}\Big|_1^\infty = 1.$$

So $1 - s\int \in [0,1]$, so

$$\zeta(s) = \frac{1}{s-1} + \left(1 - s\int\right) \in \left[\frac{1}{s-1}, \frac{1}{s-1} + 1\right].$$

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- Let's get an asymptotic formula for

$$\sum_{n \le x} \frac{1}{n} = \int_{1^-}^{x} \frac{1}{u}\,d\lfloor u \rfloor$$

$$= \int_{1^-}^{x} \frac{1}{u}\,du - \int_{1^-}^{x} \frac{1}{u}\,d\{u\}$$

$$= \log u\Big|_{1^-}^{x} - \left(\frac{1}{u}\{u\}\Big|_{1^-}^{x}\right) + \int_{1^-}^{x} \{u\}\,d\left(\frac{1}{u}\right)$$

$$= \log x - 0 - \left(\frac{\{x\}}{x} - \frac{1}{1}\right) - \int_1^x \frac{\{u\}}{u^2}\,du.$$

So

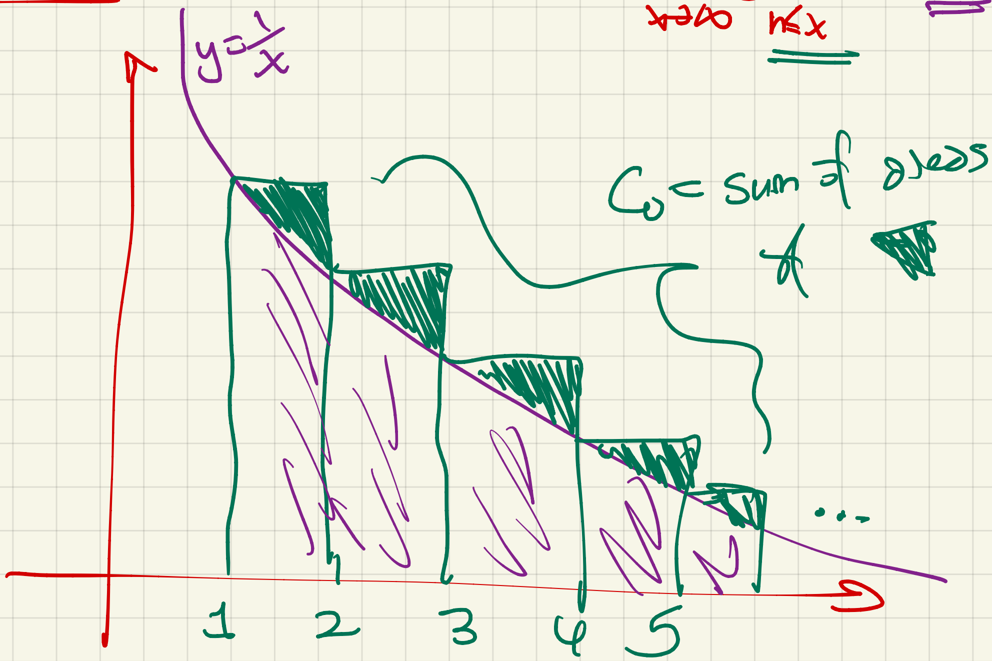
$$\sum_{n \le x} \frac{1}{n} = \log x + O\left(\frac{1}{x}\right) + 1 - \int_1^\infty \frac{\{u\}}{u^2}\,du + \int_x^\infty \frac{\{u\}}{u^2}\,du$$

$$= \log x + C_0 + O\left(\frac{1}{x}\right).$$

C_0: Euler's constant

Note: $O\left(\frac{1}{x}\right)$ is the best error we can hope for, since $\sum_{n \le x} \frac{1}{n}$ has jumps of that size.

Note 2: We see that $C_0 = \lim_{x\to\infty}\left(\sum_{n\le x}\frac{1}{n} - \log x\right)$



Note: picture shows $\frac{1}{2} < C_0 < 1$

(in fact: $C_0 \approx 0.577$).

Conjecture: C_0 is irrational

Chapter 2: Elementary estimates for arithmetic functions.

Motivating example: $0 < \frac{\phi(n)}{n} \le 1$;

What's the average value (expectation)?

- Limit to $n \le x$: $\frac{1}{\lfloor x \rfloor}\sum_{n\le x}\frac{\phi(n)}{n}$.
- If $\lim_{x\to\infty}\left(\frac{1}{\lfloor x\rfloor}\sum_{n\le x}\frac{\phi(n)}{n}\right)$ exists, we call it the mean value of $\frac{\phi(n)}{n}$.

Recall: $\phi * 1 = n$; so $\phi = n * \mu$:

$$\phi(n) = \sum_{d\mid n}\mu(d)\frac{n}{d} \iff \frac{\phi(n)}{n} = \sum_{d\mid n}\frac{\mu(d)}{d}.$$

Then:

$$\frac{1}{x}\sum_{n\le x}\frac{\phi(n)}{n} = \frac{1}{x}\sum_{n\le x}\sum_{d\mid n}\frac{\mu(d)}{d}$$

$$= \frac{1}{x}\sum_{d\le x}\frac{\mu(d)}{d}\sum_{\substack{n\le x\\ d\mid n}} 1$$

$$= \frac{1}{x}\sum_{d\le x}\frac{\mu(d)}{d}\left\lfloor\frac{x}{d}\right\rfloor = \frac{1}{x}\sum_{d\le x}\frac{\mu(d)}{d}\left(\frac{x}{d}+O(1)\right)$$

$$= \sum_{d\le x}\frac{\mu(d)}{d^2} + O\left(\frac{1}{x}\sum_{d\le x}\frac{|\mu(d)|}{d}\right)$$

$$= \sum_{d=1}^{\infty}\frac{\mu(d)}{d^2} + O\left(\sum_{d>x}\frac{|\mu(d)|}{d^2} + \frac{1}{x}\sum_{d\le x}\frac{|\mu(d)|}{d}\right)$$

$$= \frac{1}{\zeta(2)} + O\left(\sum_{d>x}\frac{1}{d^2} + \frac{1}{x}\sum_{d\le x}\frac{1}{d}\right)$$

$$= \frac{1}{\pi^2/6} + O\left(\frac{1}{x} + \frac{\log x}{x}\right).$$

$$= \frac{6}{\pi^2} + O\left(\frac{\log x}{x}\right).$$

Hence $\displaystyle\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}\frac{\phi(n)}{n} = \frac{6}{\pi^2}$

is the mean value of $\phi(n)/n$. ✓

General idea: If $f = g * 1$, then

$$\sum_{n\le x} f(n) = \sum_{n\le x}\sum_{d\mid n} g(d)$$

$$= \sum_{d\le x} g(d)\sum_{\substack{n\le x\\ d\mid n}} 1 = \sum_{d\le x} g(d)\left(\frac{x}{d}+O(1)\right).$$

Example: let's estimate $Q(x)$, the number of squarefree integers up to x:

$$Q(x) = \sum_{n\le x}\begin{cases}1, & \text{if } n \text{ is squarefree,}\\ 0, & \text{otherwise}\end{cases}$$

$$= \sum_{n\le x}\mu(n)^2.$$

Exercise: show that $\mu^2 = 1 * g$ where "$g(d) = \mu(\sqrt{d})$", that is,

$$g(d) = \begin{cases} \mu(c), & \text{if } d = c^2, \\ 0, & \text{if } d \text{ is not a square.} \end{cases}$$

So

$$Q(x) = \sum_{n \le x} \mu^2(n) = \sum_{d \le x} g(d)\left(\frac{x}{d} + O(1)\right)$$

$$\overset{d=c^2}{=} \sum_{c \le \sqrt{x}} \mu(c)\left(\frac{x}{c^2} + O(1)\right)$$

$$= x \sum_{c \le \sqrt{x}} \frac{\mu(c)}{c^2} + O\left(\sum_{c \le \sqrt{x}} |\mu(c)|\right)$$

$$= x\left(\sum_{c=1}^{\infty} \frac{\mu(c)}{c^2} + O\left(\sum_{c > \sqrt{x}} \frac{|\mu(c)|}{c^2}\right)\right) + O(\sqrt{x})$$

$$= x\left(\frac{6}{\pi^2} + O\left(\frac{1}{\sqrt{x}}\right)\right) + O(\sqrt{x})$$

$$= \frac{6}{\pi^2}x + O(\sqrt{x}).$$

"The probability that a randomly chosen integer is squarefree is $6/\pi^2$."

Terminology: Given $A \subseteq \mathbb{N}$, define its (natural) density to be

$$\delta(A) = \lim_{x \to \infty} \frac{1}{x} \#\{a \in A : a \le x\},$$

if the limit exists.

Example: $A = \{a \in \mathbb{N} : a \equiv b \pmod{c}\}$, then

$$\#\{a \in A : a \le x\} = \frac{x}{c} + O(1); \text{ so}$$

$$\delta(A) = \frac{1}{c}.$$

We showed $\delta(\{\text{squarefree numbers}\}) = \frac{6}{\pi^2}$

$$= 1/\zeta(2) = \prod_p \left(1 - \frac{1}{p^2}\right).$$

$\delta(\{\text{squarefree numbers}\}) = \frac{6}{\pi^2}$.

$= 1/\zeta(2) = \prod_p (1-\frac{1}{p^2})$.

<u>Note</u>: it's pretty easy to show that for any fixed y,

$$\delta(\{n \in \mathbb{N} : p \le y \Rightarrow p^2 \nmid n\}) = \prod_{p \le y}(1-\tfrac{1}{p^2})$$

In fact,

$$\#\{n \le x : p \le y \Rightarrow p^2 \nmid n\}$$
$$= x \prod_{p \le y}(1-\tfrac{1}{p^2}) + O_y(1)$$

But we can't take $\lim_{y\to\infty}$ because of the O_y.

<u>Leveling up the method</u>: Let $A(x) = \sum_{n \le x} a_n$ and $B(x) = \sum_{n\le x} b_n$. Define $c = a * b$ and write $C(x) = \sum_{n \le x} c_n$. Then

$$C(x) = \sum_{n \le x} \sum_{d \mid n} a_d b_{n/d}$$

$$= \sum_{d \le x} a_d \sum_{\substack{n \le x \\ d \mid n}} b_{n/d} \overset{n=md}{=} \sum_{d\le x} a_d \sum_{m \le \frac{x}{d}} b_m$$

$$= \sum_{d \le x} a_d B(\tfrac{x}{d}).$$

Previous example: $b_n \equiv 1$, so $B(x) = \lfloor x \rfloor$
$= x + O(1)$.

<u>Example</u>: Let $c_n = d(n)$, so that $c = 1 * 1$.

Examine $\sum_{n \le x} d(n)$.

$$C(x) = \sum_{d\le x} a_d B\left(\tfrac{x}{d}\right).$$

Here, $a_d = 1$ and $B\left(\frac{x}{d}\right) = \left\lfloor \frac{x}{d} \right\rfloor$, so

$$\sum_{n\le x} d(n) = \sum_{d\le x} 1\left(\frac{x}{d} + O(1)\right)$$

$$= x\sum_{d\le x}\frac{1}{d} + O\Big(\sum_{d\le x} 1\Big)$$

$$= x\left(\log x + C_0 + O\left(\tfrac{1}{x}\right)\right) + O(x)$$

$$= x\log x + O(x).$$

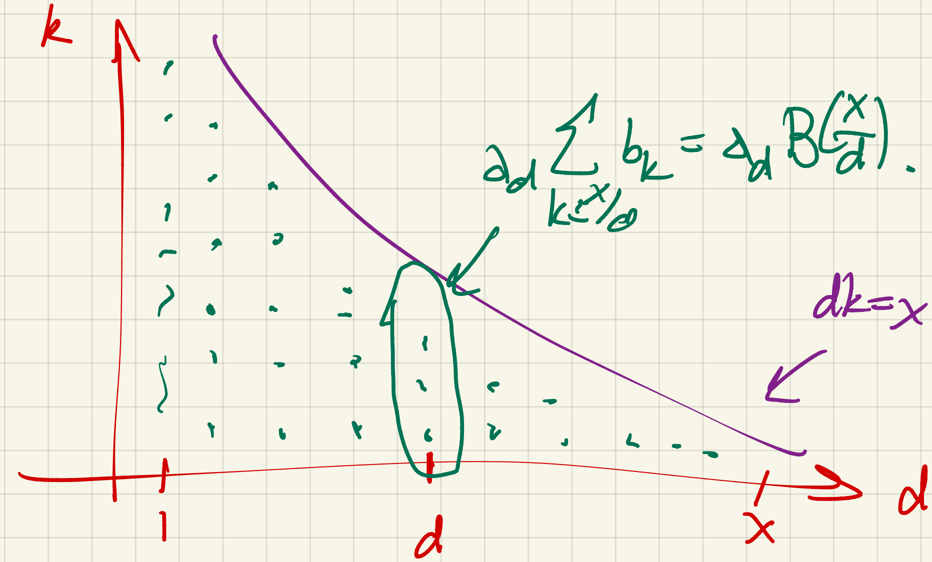
Exercise: For $x\ge 1$, $\sum_{n\le x}\log n = x\log x + O(x)$.

Thus $$\sum_{n\le x} d(n) \sim \sum_{n\le x}\log n$$

"The average order of $d(n)$ is $\log n$."

Why is the error so big?

$$\sum_{n\le x}\sum_{dk=n} a_d b_k = \sum_{d\le x} a_d B\left(\tfrac{x}{d}\right).$$



Also, $$= \sum_{k\le x} b_k A\left(\tfrac{x}{k}\right).$$

Clever idea: