Thursday, January 30

Group Work #4 today

Group Work #3 on Tuesday

Let $c = a * b$. Write $A(x) = \sum_{n \le x} a_n$ and $B(x) = \sum_{n \le x} b_n$ and $C(x) = \sum_{n \le x} c_n$, so that

$$C(x) = \sum_{n \le x} \sum_{dm = n} a_d b_m = \sum_{\substack{d, m \in \mathbb{N} \\ dm \le x}} a_d b_m.$$

On Tuesday we saw the identity

$$C(x) = \sum_{d \le x} a_d B\left(\frac{x}{d}\right).$$

But we have the more elaborate version:

For any $x, y > 0$:

$$C(x) = \sum_{d \le y} a_d B(x/d) + \sum_{m \le x/y} b_m A\left(\frac{x}{m}\right) - A(y) B(x/y).$$ [Exercise]

"Dirichlet hyperbola method"

Example, revisited: Let $c_n = d(n)$, so that $c = 1 * 1$. Note that $A(x) = B(x) = \lfloor x \rfloor$.

Then

$$\sum_{n \le x} d(n) = \sum_{d \le y} 1 \cdot B\left(\frac{x}{d}\right) + \sum_{m \le x/y} 1\, A\left(\frac{x}{m}\right) - A(y) B\left(\frac{x}{y}\right)$$

$$= \sum_{d \le y} \left(\frac{x}{d} + O(1)\right) + \sum_{m \le x/y} \left(\frac{x}{m} + O(1)\right) - (y + O(1))\left(\frac{x}{y} + O(1)\right)$$

$$= x \sum_{d \le y} \frac{1}{d} + O(y) + x \sum_{m \le x/y} \frac{1}{m} + O\left(\frac{x}{y}\right) - x + O\left(\frac{x}{y}\right) + O(y) + O(1)$$

$$= x \sum_{d \le y} \frac{1}{d} + x \sum_{m \le x/y} \frac{1}{m} - x + O\left(y + \frac{x}{y} + \cancel{1}\right).$$

assume $x, y \ge 1$.

Using $\sum_{d \le y} \frac{1}{d} = \log y + C_0 + O\left(\frac{1}{y}\right)$:

$$\sum_{n \le x} d(n) = x\left(\log y + C_0 + O\left(\frac{1}{y}\right)\right) + x\left(\log \frac{x}{y} + C_0 + O\left(\frac{1}{x/y}\right)\right) - x + O\left(y + \frac{x}{y}\right)$$

$$= x \log x + (2C_0 - 1)x + O\left(y + \frac{x}{y}\right).$$

Choose $y = \sqrt{x}$ to minimize the error term:

$$\sum_{n \le x} d(n) = x \log x + (2C_0 - 1)x + O(\sqrt{x}).$$

Reading: "Dirichlet divisor problem"

Recall the von Mangoldt Lambda-function

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k, \\ 0, & \text{otherwise.} \end{cases}$$

We know $\log = \Lambda * 1$.

$$-\zeta'(s) = -\frac{\zeta'}{\zeta}(s) \cdot \zeta(s)$$

Idea: use info about $\log$ to get info about Λ.

Specific formulas:

- $\sum_{n \le x} \log n = \sum_{d \le x} \Lambda(d) \left\lfloor \frac{x}{d} \right\rfloor$.
- Exercise (compare to $\int_1^x \log t \, dt$):

$$\sum_{n \le x} \log n = x \log x - x + O(\log x).$$