# MATH 539
# Analytic Number Theory
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Motivating questions (vague forms):

- What is the "probability" that a randomly chosen integer has a particular property?
  - prime
  - squarefree (not divisible by any $d^2$ other than $1^2$)
  - odd # of number factors
- Given a multiplicative or additive function like
  - $d(n)$, the number of (positive) divisors of $n$;
  - $\sigma(n)$, the sum of those divisors;
  - Euler phi-function $\phi(n)$;
  - Möbius function $\mu(n)$;
  - # of prime factors of $n$ (with or without multiplicity);

  What is its distribution on "randomly chosen integers"?
  - minimum / maximum
  - average values
  - typical values

To make these questions rigorous:

- Let $x \ge 1$ be a parameter.
- Answer any of the above questions when $n$ is chosen uniformly from $\{1, 2, \ldots, \lfloor x \rfloor\}$. – answer depends on $x$
- Take limits as $x \to \infty$ (possibly normalize)

Example: The prime number theorem.

Let $\pi(x) = \#\{\text{primes} \le x\}$. We will show that $\pi(x) \approx x/\log x$. More precisely,

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1.$$

(Note: in this course, log means natural log.)

Plan for January:

- A little notation and a technical tool
- Fundamentals of "Dirichlet series"
$$\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$
- Properties of Riemann zeta function
$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$ and consequences for finite sums $\sum_{1\le n\le x} 1/n^p$.
- "Elementary" (not using complex analysis) distribution statements

Then back to Dirichlet series and $\zeta(s)$.

Notation for error terms:
(p. xvii / Section 1 of MV)

Let $g(x)$ be a nonnegative function.

Notation: The expression $O(g(x))$ denotes an unspecified function, $u(x)$, such that $|u(x)| \le C g(x)$ for some constant $C$.
(on whatever domain we're considering)

Example 1: Let's show that
$$e^{2x} - 1 = 2x + O(x^2) \quad \text{for } -1\le x\le 1.$$

Intuition: tangent line to $y = e^{2x}-1$ at $x=0$ is $y=2x$; "fits quadratically"

**Example 1:** Let's show that
$$e^{2x}-1 = 2x + O(x^2) \text{ for } -1 \le x \le 1.$$

**Solution:** The function $e^{2z}-1-2z$ is entire and has a double zero at $z=0$.
Therefore $\dfrac{e^{2z}-1-2z}{z^2}$ has a removable singularity at $z=0$, and thus extends to an entire function $h(z)$.
Define $C = \max\{|h(z)| : |z| \le 1\}$.

Then
$$|h(z)| \le C$$
$$|e^{2z}-1-2z| \le C|z|^2$$
$$e^{2z}-1-2z = O(|z|^2) \text{ on } \{|z| \le 1\}$$
Thus
$$e^{2x}-1-2x = O(|x|^2) \text{ on } [-1,1]$$
$$e^{2x}-1 = 2x + O(x^2).$$

**Exercise:** Show that $\sqrt{x+1} = \sqrt{x} + O\left(\frac{1}{\sqrt{x}}\right)$ for $x \in [1,\infty)$. (Hint: Mean value theorem)

**Notation:** $f(x) \ll g(x)$ is a synonym for $f(x) = O(g(x))$. [TeX: \ll]

**Examples:**
- $e^{2z}-1-2z \ll |z|^2$ for $\{|z| \le 1\}$.
- $2x \ll x$ for $x \in [0,\infty)$.
- $x \ll x^2$ for $x \in [1,\infty)$, but $x \not\ll x^2$ for $x \in (0,1]$.

**Question:** Suppose $f_1(x) \ll g_1(x)$ and $f_2(x) \ll g_2(x)$.
- Is $f_1(x)f_2(x) \ll g_1(x)g_2(x)$?

YES: $|f_1(x)| \le C_1 g_1(x)$ and $|f_2(x)| \le C_2 g_2(x)$.
Multiplying: $\underbrace{|f_1(x)||f_2(x)|}_{|f_1(x)f_2(x)|} \le C_1C_2\, g_1(x)\, g_2(x)$.

What about: is $\frac{f_1(x)}{f_2(x)} \ll \frac{g_1(x)}{g_2(x)}$?

No – we would need a lower bound on $f_2$, not an upper bound.

**Exercise:** If $f_1(x) \ll g_1(x)$ and $f_2(x) \ll g_2(x)$, show that $f_1(x) + f_2(x) \ll \max\{g_1(x), g_2(x)\}$.

**Exercise:** Let $f(x)$, $g(x)$ be continuous on $[0, \infty)$, with $g(x) > 0$. Suppose that $f(x) \ll g(x)$ on $[539, \infty)$. Show that $f(x) \ll g(x)$ on $[0, \infty)$. (Hint: $\frac{|f(x)|}{g(x)}$ is continuous on $[0, 539]$.)

– Here we can say "$f(x) \ll g(x)$ as $x \to \infty$"

– Similarly: $e^{2z} - 1 - 2z \ll |z|^2$ "for $z$ near $0$".

**Notation:** $f(x) \sim g(x)$ means $\lim \frac{f(x)}{g(x)} = 1$, "$f$ is asymptotic to $g$"

(often $\lim_{x \to \infty}$, or from context)

Also, $f(x) = o(g(x))$ means $\lim \frac{f(x)}{g(x)} = 0$.

**Example:** The prime number theorem is $\pi(x) \sim x/\log x$. Also,

$$\pi(x) = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right)$$
$$= (1 + o(1)) \frac{x}{\log x}.$$

Example: Let $\alpha, \beta \in \mathbb{R}$, and $x \in \mathbb{R}$.

- $\sin(\alpha x) + \cos(\beta x) \ll 1$. (triangle inequality)

- $\alpha \sin x + \beta \cos x \ll |\alpha| + |\beta|$.

If we write $\alpha \sin x + \beta \cos x \ll 1$, this would depend on whether $\alpha$ and $\beta$ were restricted or not. But if we're okay with the upper bound depending on $\alpha$ and $\beta$, we can write

$$\alpha \sin x + \beta \cos x \ll_{\alpha, \beta} 1.$$

The "implicit constant" can depend on $\alpha$ and $\beta$, but not $x$.

Exercise: For any $A, \varepsilon > 0$, show that

$$(\log x)^A = o_{A,\varepsilon}(x^\varepsilon) \text{ on } [1, \infty).$$

Hint: use l'Hôpital on $\dfrac{\log x}{x^{\varepsilon/A}}$.