Thursday, January 9

(Group Work #1 today)

Riemann-Stieltjes integrals (Appendix A of MV)

– for us, this is just a helpful tool

Review Riemann integral:

- partition $[c,d]$: $\underline{x} = \{x_0, x_1, \dots, x_N\}$ with $x_0 = c$ and $x_N = d$, and $x_0 < x_1 < \dots < x_N$.

  mesh size $m(\underline{x}) = \max_{1 \le i \le N} (x_i - x_{i-1})$.

  sample points $\xi_i \in [x_{i-1}, x_i]$.

Then define $$\int_c^d f(x)\,dx = \lim_{m(\underline{x}) \to 0} \sum_{i=1}^{N} f(\xi_i)(x_i - x_{i-1})$$

if the limit exists.

Theorem: If f is bounded and piecewise* continuous on $[c,d]$, then $\int_c^d f(x)\,dx$ exists.

* piecewise means finitely many pieces

Un-example: if $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}, \end{cases}$

then $\int_c^d f(x)\,dx$ does not exist.

Riemann-Stieltjes integral: given two functions f, g, define

$$\int_c^d f(x)\,dg(x) = \lim_{m(\underline{x}) \to 0} \sum_{i=1}^{N} f(\xi_i)(g(x_i) - g(x_{i-1}))$$

if the limit exists.

Riemann-Stieltjes integral: given two functions f, g, define

$$\int_c^d f(x)\, dg(x) = \lim_{m(\Delta)\to 0} \sum_{i=1}^{N} f(\xi_i)(g(x_i)-g(x_{i-1}))$$

if the limit exists.

Theorem (A.1 and A.2) If f "has bounded variation" on $[c,d]$ and g is continuous on $[c,d]$, or vice versa, then

$$\int_c^d f(x)\, dg(x) \text{ exists.}$$

- Any bounded, piecewise monotone function is of bounded variation.

Un-example: $\sin\left(\frac{1}{x}\right)$ is not of bounded variation on $(0,1)$.

Example:- Given any sequence $(a_1, a_2, \ldots)$, define $A(x) = \sum_{n\le x} a_n$. Then on any $[c,d]$, $A(x)$ is

- bounded
- piecewise constant
- bounded variation

- $\int_c^d A(x)\,dx$ exists
- $\int_c^d A(x)\, dg(x)$ exists for all continuous g.

Three key facts about Riemann-Stieltjes integrals:

$\boxed{\text{I}}$ Let $A(x) = \sum_{n \le x} a_n$. If $f(x)$ is any continuous function, then

$$\int_c^d f(x)\, dA(x) = \sum_{c < n \le d} a_n f(n)$$

$\boxed{\text{II}}$ Integration by parts (Theorem A.2):

$$\int_c^d f(x)\, dg(x) = f(x)g(x)\Big|_c^d - \int_c^d g(x)\, df(x)$$

$$= \big(f(d)g(d) - f(c)g(c)\big) - \int_c^d g(x)\, df(x).$$

$\boxed{\text{III}}$ "Un-Stieltjesification" (Thm A.3): If g is Riemann integrable, and f is continuously differentiable, then

$$\underbrace{\int_c^d g(x)\, df(x)}_{\text{R-S}} = \underbrace{\int_c^d g(x) f'(x)\, dx}_{\text{Riemann}}.$$

Example: With $A(x) = \sum_{n \le x} a_n$, for any $\alpha \in [0,1)$ we have

$$\sum_{n \le x} \frac{a_n}{n} = \sum_{n \le x} a_n \frac{1}{n}$$

$$\boxed{\text{I}} \quad = \int_\alpha^x \frac{1}{t}\, dA(t)$$

$$\boxed{\text{II}} \quad = \frac{A(t)}{t}\Big|_\alpha^x - \int_\alpha^x A(t)\, d\frac{1}{t}$$

$$\boxed{\text{III}} \quad = \frac{A(t)}{t}\Big|_\alpha^x - \int_\alpha^x A(t)\left(-\frac{1}{t^2}\right) dt.$$

take limit as $\alpha \nearrow 1$.

$$\frac{A(x)}{x} - \int_1^x A(t)\left(-\frac{1}{t^2}\right) dt$$

Checklist for all group discussions:

- Regular polygon
- Proactive balance
- Respect for differences