Tuesday, March 11

Group Work #7 on Thursday

Definition: For $u>0$, define

$$\theta(u) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 u}.$$

It satisfies the functional equation

$$\theta(u) = u^{-1/2}\theta\left(\frac{1}{u}\right).$$

(Proof: Poisson summation).

If we set $\theta_+(u) = \sum_{n=1}^{\infty} e^{-\pi n^2 u}$ $(u>0)$, then $\theta(y) = 1 + 2\theta_+(y)$. The functional equation becomes:

$$\theta_+(1/u) = -\frac{1}{2} + \frac{1}{2}\sqrt{u} + \sqrt{u}\,\theta_+(u) \quad (u>0) \qquad (*)$$

Exercise: $\theta_+(u) \ll e^{-\pi u}$ for $u \geq 1$.

It follows that $\theta_+(u) \sim \frac{1}{2\sqrt{u}}$ as $u \to 0+$.

Fun calculation with a secret punchline

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\,dx \qquad (\sigma>0)$$

Changing x to $\pi n^2 x$, and s to $s/2$:

$$\pi^{-s/2} n^{-s}\,\Gamma(s/2) = \int_0^\infty x^{s/2-1} e^{-\pi n^2 x}\,dx$$

Summing over all $n \geq 1$: $(\sigma>1)$

$$\pi^{-s/2}\zeta(s)\Gamma(s/2) = \sum_{n=1}^{\infty}\int_0^\infty x^{s/2-1} e^{-\pi n^2 x}\,dx$$

(by absolute convergence)

$$= \int_0^\infty x^{s/2-1}\,\theta_+(x)\,dx. \qquad (**)$$

Note that $\int_0^1 x^{s/2-1}\theta_+(x)\,dx$

$$= \int_\infty^1 \left(\frac{1}{x}\right)^{s/2-1}\theta_+\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\,dx\right)$$

$$= \int_1^\infty x^{-s/2-1}\left(-\frac{1}{2} + \frac{1}{2}\sqrt{x} + \sqrt{x}\,\theta_+(x)\right)dx$$

$$= -\frac{1}{2}\int_1^\infty x^{-s/2-1} + \frac{1}{2}\int_1^\infty x^{-s/2-1/2}\,dx + \int_1^\infty x^{-s/2-1/2}\theta_+(x)\,dx$$

$$= -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty x^{-s/2-1/2}\theta_+(x)\,dx.$$

Therefore, from (***),

$$\pi^{-s/2}\zeta(s)\Gamma(s/2) = \left(\int_0^1 + \int_1^\infty\right) x^{s/2-1}\theta_+(x)\,dx$$

$$= -\frac{1}{s} - \frac{1}{1-s} + \int_1^\infty \theta_+(x)\left(x^{\frac{s}{2}-1} + x^{\frac{1-s}{2}-1}\right)dx.$$

Two big observations:

- The right-hand side is meromorphic for all $s \in \mathbb{C}$ (the integral converges for all s). — Hence $\zeta(s)$ is meromorphically continued to $\mathbb{C}$.
- The right-hand side is invariant under changing s to $1-s$. !!

Functional equation (Riemann) (Corollary 10.3)

Define the "completed zeta-function"

$$\xi(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma(s/2)\pi^{-s/2}.$$

Then $\xi(s)$ is an entire function, and $\xi(s) = \xi(1-s)$.

We can derive an asymmetric form of this functional equation as follows:

$$\cancel{\tfrac{1}{2}s(s-1)}\,\zeta(s)\,\Gamma\!\left(\tfrac{s}{2}\right)\pi^{-s/2}$$

$$= \cancel{\tfrac{1}{2}(1-s)(1-s-1)}\,\zeta(1-s)\,\Gamma\!\left(\tfrac{1-s}{2}\right)\pi^{-(1-s)/2}$$

$$\zeta(s) = \zeta(1-s)\,\pi^{s-\frac{1}{2}}\,\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\,\frac{\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}.$$

Using the duplication formula on top and the reflection formula on bottom:

$$\zeta(s) = \zeta(1-s)\,\pi^{s-\frac{1}{2}}\,\frac{\sqrt{\pi}\,2^{1-2\left(\frac{1-s}{2}\right)}\,\Gamma\left(2\,\frac{1-s}{2}\right)}{\pi/\sin\left(\pi\frac{s}{2}\right)}$$

Simplifying:

Corollary 10.4: For $s \in \mathbb{C}$,

$$\zeta(s) = \zeta(1-s)\,\pi^{s-1}\,2^{s}\,\Gamma(1-s)\sin\frac{\pi s}{2}.$$

Consequences: (think $\sigma \le 0$)

- $\zeta(-2)=0$, $\zeta(-4)=0$, $\zeta(-6)=0$, ...
  - called the "trivial zeros"

But otherwise $\zeta(s) \neq 0$ for $\sigma \le 0$.

- From Suggested Problems #2, we know how to work out exact values of $\zeta(2)$, $\zeta(4)$, $\zeta(6)$, ... For example,

$$\zeta(12) = 691\pi^{12}/638{,}512{,}875.$$

Thus the functional equation allows us to compute $\zeta(-1)$, $\zeta(-3)$, $\zeta(-5)$, ...

Examples:
$$\zeta(-11) = \frac{691\pi^{12}}{638512875}\,\pi^{-11-1}\,2^{-11} \cdot \underbrace{\Gamma(1-(-11))}_{11!}\,\underbrace{\sin\left(-11\pi/2\right)}_{1} = \frac{691}{32{,}760}.$$

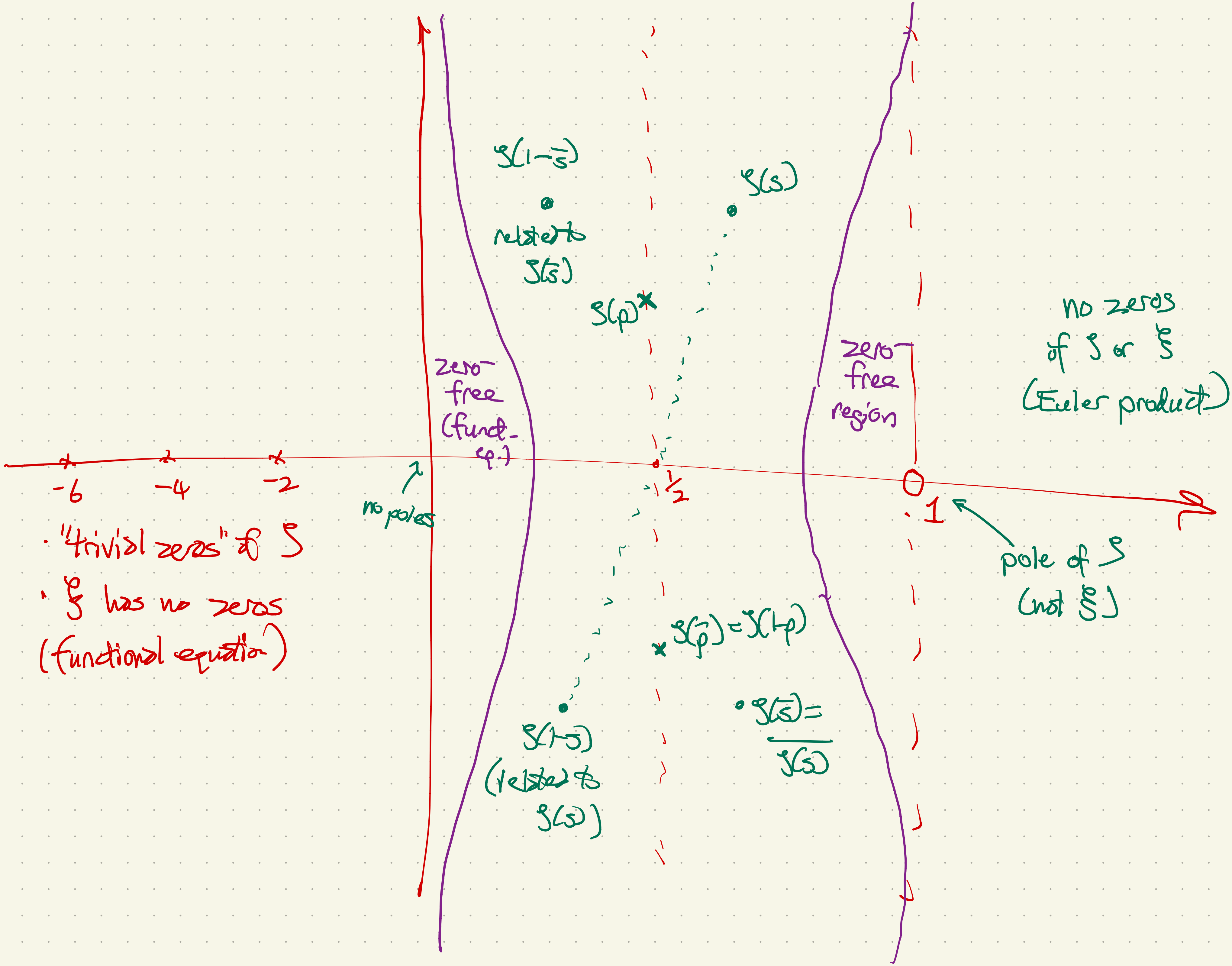
→ related to "Bernoulli numbers".

Exercise: Show that $\zeta(0) = -\frac{1}{2}$.

Corollary 10.5: Uniformly for $|\sigma| \le A$ and $|t| \ge 1$,

$$|\zeta(s)| \asymp_A \tau^{\frac{1}{2}-\sigma}\,|\zeta(1-s)|.$$

Schwarz reflection principle: if $f(z)$ is analytic and real for $z \in \mathbb{R}$, then $f(\bar{z}) = \overline{f(z)}$.



ξ(1−s̄)
related to ξ(s̄)
ξ(s)
ξ(ρ)
zero-free (funct.-eq.)
zero-free region
no zeros of ζ or ξ (Euler product)
−6
−4
−2
no poles
1/2
0
1
pole of ζ (not ξ)
"trivial zeros" of ζ
ξ has no zeros (functional equation)
ξ(ρ̄)=ξ(1−ρ)
ξ(s̄) = $\overline{ξ(s)}$
ξ(1−s) (related to ξ(s))

Inside the critical strip $\{s \in \mathbb{C} : 0 < \sigma < 1\}$, lie the "nontrivial zeros" of $\zeta(s)$ (which are all the zeros of $\xi(s)$). They come in pairs (on the critical line) or quadruplets.

The Riemann Hypothesis is the assertion (still unproved) that all nontrivial zeros of $\zeta(s)$ have real part exactly equal to $\frac{1}{2}$.

- verified for the first $\approx 1.2 \times 10^{13}$ zeros (up to height 3×10^{12}).

Section 10.2: Hadamard products

Lemma 10.11: Let $f(z)$ be an entire function, $f(0) \neq 0$. Suppose there exists $\theta < 2$ such that

$$\max_{|z|=R} |f(z)| \ll \exp(R^\theta) \text{ for all } R > 0.$$

Then there exists $A, B \in \mathbb{C}$ such that

$$f(z) = e^{A+Bz} \prod_{\substack{w \in \mathbb{C} \\ f(w)=0}} \left(1 - \frac{z}{w}\right) e^{z/w}.$$