Thursday, March 13

Group Work #7 today

Suggested Problems #4 posted today

RECALL THAT WE STATED:

Lemma 10.11: Let $f(z)$ be an entire function, $f(0) \neq 0$. Suppose there exists $\theta < 2$ such that
$$\max_{|z|=R} |f(z)| \ll \exp(R^\theta) \text{ for all } R>0.$$
Then there exists $A, B \in \mathbb{C}$ such that
$$f(z) = e^{A+Bz} \prod_{\substack{w\in\mathbb{C} \\ f(w)=0}} \left(1-\frac{z}{w}\right)e^{z/w}.$$

Outline of proof:

- Let $N(R) = \#\{w \in \mathbb{C} : |w| \le R,\ f(w)=0\}$. Jensen's inequality (Lemma 6.1) implies that $N(R) \ll R^\theta$. (cool!)
- $\log \prod_w \left(1-\frac{z}{w}\right)e^{z/w} = \sum_w \left(\log\left(1-\frac{z}{w}\right) + \frac{z}{w}\right)$
  $\ll \sum_w \left|\frac{z}{w}\right|^2$ (Taylor approximation); and
  $\sum_w \frac{1}{|w|^2} = \int_0^\infty \frac{1}{u^2}\, dN(u)$, which converges (to $\int_0^\infty \frac{2N(u)}{u^3}\,du$) since $N(u) \ll u^\theta$.
- Then $h(z) = \frac{f(z)}{f(0)}\left(\prod_w \left(1-\frac{z}{w}\right)e^{z/w}\right)^{-1}$ is entire and nonvanishing; so $\log h(z)$ is entire.
- One can show that $\max_{|z|=R} \operatorname{Re} \log h(z) = \max_{|z|=R} \log|h(z)| \ll R^\theta \log R$. By Borel–Carathéodory (Lemma 6.2), $|\log h(z)| \ll |z|^\theta \log|z|$.

Hence $\log h(z) = A + Bz$ for some A, B. //

Recall $\xi(s) = \frac{1}{2} s(s-1) \zeta(s) \Gamma(s/2)$.

Theorem 10.12: $\xi(s) = \frac{1}{2} e^{Bs} \prod_{\rho} (1-\frac{s}{\rho}) e^{s/\rho}$

for some $B \in \mathbb{C}$, where the product runs over all zeros of ξ (equivalently, over the nontrivial zeros ρ of ζ — the zeros inside the critical strip $\{0 < \sigma < 1\}$).

Proof sketch: First assume $\sigma \geq \frac{1}{2}$. Then $\zeta(s) \ll \tau^{1/2}$ by Corollary 1.17, and $\Gamma(z) \ll e^{|z| \log |z|}$ by Stirling's formula (Theorem C.1).

Hence* $\xi(s) \ll e^{|s| \log |s|} \ll e^{|s|^{1.5}}$, so we can apply Lemma 10.11. Calculate that $\xi(0) = \frac{1}{2}$.

*for $\sigma \geq \frac{1}{2}$; but the functional equation $\xi(1-s) = \xi(s)$ implies it for $\sigma \leq \frac{1}{2}$ also. //

Corollary 10.14: (a) $\frac{\xi'}{\xi}(s) = B + \sum_{\rho} (\frac{1}{s-\rho} + \frac{1}{\rho})$

(direct from Theorem 10.12)

(b) $\frac{\zeta'}{\zeta}(s) = B + \frac{1}{2} \log \pi - \frac{1}{s-1} - \frac{1}{2} \frac{\Gamma'}{\Gamma}(\frac{s}{2}+1)$
$+ \sum_{\rho} (\frac{1}{s-\rho} + \frac{1}{\rho})$.

(direct from definition of ξ)

(c) $B \underset{(1)}{=} -\frac{1}{2} \sum_{\rho} (\frac{1}{1-\rho} + \frac{1}{\rho}) \underset{(2)}{=} -\sum_{\rho} \mathrm{Re}(\frac{1}{\rho})$

$\underset{(3)}{=} -\frac{C_0}{2} - 1 + \frac{1}{2} \log 4\pi \approx -0.0231$.

(1): Compare $\frac{\xi'}{\xi}(0)$ and $\frac{\xi'}{\xi}(1)$ from (a), and use FE.

(2) $-\frac{1}{2} \sum_{\rho} (\frac{1}{1-\rho} + \frac{1}{\rho}) = -\frac{1}{2} \sum_{\rho} \mathrm{Re}(\frac{1}{1-\rho} + \frac{1}{\rho})$

$= -\frac{1}{2} (\sum_{\rho} \mathrm{Re}(\frac{1}{1-\rho}) + \sum_{\rho} \mathrm{Re}(\frac{1}{\rho})) = -\sum_{\rho} \mathrm{Re}(\frac{1}{\rho})$.

(3) take $s=0$ in part (b).