Tuesday, March 18
Group Work #8 on Thursday

RECALL

- $\xi(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma(s/2)$.
- $\xi(s) = \frac{1}{2}e^{Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}$, where $\prod_{\rho}$ runs over the nontrivial zeros of $\zeta(s)$

Observation: $\xi(s)$ has infinitely many zeros.

Proof: If not, then $\xi(s) = e^{Cs} \times$ polynomial where $C = B + \sum_{\rho} 1/\rho$. Therefore $\log|\xi(s)| \ll |s|$ for all $s \in \mathbb{C}$. But as $s \to \infty$ through real values, $\xi(s) \gg$ polynomial $\times \Gamma(s/2)$, so $\log \xi(s) \gg s \log s$ by Stirling's formula (Theorem C.1). //

Definition: $N(T) = \#\{\rho = \beta + i\gamma : \zeta(\rho) = 0,\ 0 < \beta < 1,\ 0 \le \gamma \le T\}$ if T is not an ordinate of a zero of $\zeta(s)$; if it is, $N(T) = \frac{1}{2}(N(T-) + N(T+))$.

Theorem 10.13: For $T \ge 2$,

$$N(T+1) - N(T) \ll \log T.$$

(proof uses Jensen's formula)

Exercise: Prove that $\sum_{\rho} \mathrm{Re}\left(\frac{1}{\rho}\right)$ converges (absolutely).

Lemma 12.2: For each $T \ge 2$, there exists $T_1 \in [T, T+1]$ such that $\frac{\zeta'}{\zeta}(\sigma + iT_1) \ll \log^2 T_1$ for all $-1 \le \sigma \le 2$.

(x, y) ordinate

abscissa

Lemma 12.2: For each $T \geq 2$, there exists $T_1 \in [T, T+1]$ such that $\frac{\zeta'}{\zeta}(\sigma + iT_1) \ll \log^2 T_1$ for all $-1 \leq \sigma \leq 2$.

Proof: Recall Lemma 6.4: $\frac{\zeta'}{\zeta}(s) =$

$$\sum_{\rho}{}' \frac{1}{s-\rho} + O(\log \tau).$$

$|\rho - (\frac{3}{2} + it)| \leq 5/6$. In Lemma 6.4, we had the restriction $\sigma \geq 5/6$, but we can now extend to $\sigma \geq -1$ (Lemma 12.1).

Since there are $\ll \log T$ ordinates of zeros in $[T, T+1]$, we can choose $T_1 \in [T, T+1]$ with $|T_1 - \gamma| \gg 1/\log T$ for each ordinate γ. Then

$$\sum_{\text{nearby } \rho} \frac{1}{s-\rho} \ll \sum_{\rho} \frac{1}{|T_1 - \gamma|} \ll \sum_{\text{nearby } \rho} \log T \ll \log T \cdot \log T. \quad //$$

Since $\log \zeta(s)$ is not entire, we need to say what we mean by $\arg \zeta(s) = \operatorname{Im} \log \zeta(s)$.

Note $\zeta(s) \to 1$ as $\sigma \to \infty$, so "anchor" the log by saying $\log \zeta(s) \to 0$ as $\sigma \to \infty$.

We define $\arg \zeta(\sigma + it)$ to be continuous (in σ) on each horizontal line if t is not the ordinate of a zero of $\zeta(s)$; if it is, define $\arg \zeta(\sigma+it) = \frac{1}{2}(\arg \zeta(\sigma + it-) + \arg \zeta(\sigma + it+))$.

Lemma 12.3: $\arg \zeta(\sigma + it) \ll \log \tau$ uniformly for $-1 \leq \sigma \leq 2$.

Proof: $\arg \zeta(\sigma+it) = \operatorname{Im} \log \zeta(\sigma + it) =$

$$\operatorname{Im}\left(\log \zeta(2+it) - \int_\sigma^2 \frac{\zeta'}{\zeta}(x+it)\,dx\right)$$

$$= O(1) - \sum_{\text{nearby } \rho} \int_\sigma^2 \operatorname{Im} \frac{1}{x+it-\rho}\,dx + \int_\sigma^2 (\log \tau)\,dx$$

$$\overset{!}{=} \sum_{\rho} \left(\arctan \frac{\sigma - \beta}{t-\gamma} - \arctan \frac{2-\beta}{t-\gamma}\right) + O(\log \tau)$$

$$\ll \log \tau \cdot \left(\frac{\pi}{2} + \frac{\pi}{2}\right) \ll \log \tau.$$

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Lemma 12.4: If $\sigma \leq -1$ and $|s-(-2k)| \geq \frac{1}{4}$ for all $k \in \mathbb{N}$, then

$$\frac{\zeta'}{\zeta}(s) \ll \log(|s|+1).$$

(Proof: Functional equation and Theorem C.1)

Recall $\psi_0(x) = \sum_{n \leq x}{}' \Lambda(n)$

Theorem 12.5: For $x, T \geq 2$,

$$\psi_0(x) = x - \sum_{\substack{\rho \\ |\gamma| \leq T}} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log(1-x^{-2})$$

$$+ O\left(\min\left\{1, \frac{x}{T\langle x\rangle}\right\}\log x + \frac{x}{T}\log^2(xT)\right).$$

$\langle x \rangle$ = distance from x to the nearest prime power $\neq x$

Remark: Letting $T \to \infty$ yields my favourite formula in all of mathematics:

$$\underset{\uparrow \text{ EXACTLY!}}{\psi_0(x) =} x - \sum_{\rho} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log(1-x^{-2}).$$

Exercise: $\dfrac{x^\rho}{\rho} + \dfrac{x^{\bar\rho}}{\bar\rho} =$

$$= \frac{2}{|\rho|^2} x^\beta \left(\gamma \underset{\sin(\gamma \log x)}{\cancel{\sin(\log x)}} + \beta \underset{\cos(\gamma\log x)}{\cancel{\cos(\log x)}}\right).$$

When γ is large, $\approx \dfrac{2}{\gamma} x^\beta \underset{\sin(\gamma \log x)}{\cancel{\sin(\log x)}}$.

So Theorem 12.5 (the "explicit formula") contains an infinite sum of oscillating terms, and the biggest terms (as functions of x) correspond to the largest values of β.

Proof of Theorem 12.5: Again start with Perron's formula, with $\sigma_0 = 1 + 1/\log x$:

$$\psi_0(x) = \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0+iT} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds$$
$$+ O\Big(\sum_{\substack{\frac{x}{2}<n<2x \\ n\neq x}} \Lambda(n)\min\Big\{1, \frac{x}{T|x-n|}\Big\} + \frac{x\log x}{T}\Big).$$

On February 27, we bounded the two summands closest to x by 1, and the rest were bound by $\frac{x\log^2 x}{T}$ ($\log x$ from $\Lambda(n)$, and another $\log x$ from the harmonic sum). Here, we keep the $\min\{\}$ for those closest summands, and note that $|x-n| \geq \langle x \rangle$ since $\Lambda(n)$ is supported on prime powers.

So we get
$$\psi_0(x) = \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds + O\Big(\min\Big\{1, \frac{x}{T\langle x\rangle}\Big\}\log x + \frac{x\log^2 x}{T}\Big).$$

Let $K \in \mathbb{N}$ be odd. Consider the rectangle R with corners $\sigma_0 \pm iT_1$ and $-K \pm iT_1$, where T_1 is from Lemma 12.2.

By the residue theorem we get residues at $s=1$, nontrivial zeros of ζ, $s=0$, and trivial zeros of ζ:

$$\frac{1}{2\pi i}\oint_R -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds = \frac{x^1}{1} - \sum_{\substack{\rho \\ |\gamma|\leq T}} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0)x^0 - \sum_{k=1}^{(K-1)/2}\frac{x^{-2k}}{-2k}$$
$$= x - \sum_{\substack{\rho \\ |\gamma|\leq T}}\frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\Big(\log(1-x^{-2}) - \underbrace{\sum_{k=(K+1)/2}^{\infty}\frac{x^{-2k}}{-2k}}\Big)$$

It only remains to estimate the rest of the contour.

Exercise: $\ll \frac{1}{Kx^K}$.

- top edge, near the critical strip: use Lemma 12.2

$$\int_{-1+iT_1}^{\sigma_0+iT_1} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds \ll \int_{-1}^{\sigma_0} (\log^2 T)\frac{x^{\sigma}}{T}\,d\sigma$$

$$\ll \frac{x\log^2 T}{T}.$$

- top edge, $\sigma \le -1$: use Lemma 12.4

$$\int_{-K+iT_1}^{-1+iT_1} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds \ll \int_{-K}^{-1} (\log T)\frac{x^{\sigma}}{T}\,d\sigma$$

$$\ll \frac{\log T}{T\log x}\left(x^{-1}-x^{-K}\right).$$

- left edge: again Lemma 12.4:

$$\int_{-K-iT_1}^{-K+iT_1} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds \ll \int_{-T_1}^{T_1} \log(K+T)\frac{x^{-K}}{K}\,dt$$

$$\ll \frac{T\log(K+T)}{Kx^K}.$$

Take $K\to\infty$; two error terms disappear.

Exercise: From Theorem 12.5 and the zero-free region, deduce

$$\psi(x) = x + O\left(xe^{-c\sqrt{\log x}}\right).$$

Theorem 13.1: Assuming the Riemann hypothesis, $\psi(x) = x + O(\sqrt{x}\log^2 x)$.

Proof: Start from $\psi(x) = x - \sum_{\substack{\rho \\ |\gamma|<T}} \frac{x^\rho}{\rho}$

$+\,O\left(\log x + \frac{x\log^2 Tx}{T}\right)$. On RH, the sum is $x^{\frac12}\sum_{\substack{\rho\\|\gamma|\le T}} \frac{x^{i\gamma}}{\rho} \ll x^{\frac12}\sum_{\substack{\rho\\|\gamma|\le T}}\frac{1}{\gamma}$. By Theorem 10.13,

$$\sum_{\substack{\rho\\|\gamma|<T}}\frac{1}{\gamma} = \sum_{k=1}^{T}\left(N(k+1)-N(k)\right)\frac{1}{k}$$

$$\ll \sum_{k=1}^{T}\frac{\log k}{k} \ll \log^2 T.$$

Take $T=\sqrt{x}$ or $T=x$ ...

$x^{\frac12}\log^2 T \ll x^{\frac12}\log^2 x$ is the worst error term.