Thursday, March 20

Group Work #8 today

RECALL

Theorem 13.1: Assuming the Riemann hypothesis, $\psi(x) = x + O(\sqrt{x}\log^2 x)$.

Exercise: Assuming the Riemann hypothesis, show that $\theta(x) = x + O(\sqrt{x}\log^3 x)$ and $\pi(x) = li(x) + O(\sqrt{x}\log x)$.

Proposition: If $\psi(x) = x + O_\varepsilon(x^{\frac{1}{2}+\varepsilon})$ for all $\varepsilon > 0$, then the Riemann hypothesis holds.

Proof: Consider

$$\alpha(s) = \sum_{n=1}^{\infty} (\Lambda(n) - 1) n^{-s} = -\frac{\zeta'}{\zeta}(s) - \zeta(s),$$

and let $A(x) = \sum_{n\le x} (\Lambda(n) - 1) = \psi(x) - x + O(1)$.

$$\ll_\varepsilon x^{\frac{1}{2}+\varepsilon}.$$

By Theorem 1.3,

$$\sigma_c = \limsup_{x\to\infty} \frac{\log|A(x)|}{\log x}$$

$$\le \limsup_{x\to\infty} \frac{x^{\frac{1}{2}+\varepsilon} + \log C_\varepsilon}{\log x} = \frac{1}{2} + \varepsilon.$$

Thus $\sigma_c \le \frac{1}{2}$; so $\alpha(s)$ converges to an analytic function for $\sigma > \sigma_c = \frac{1}{2}$. Since $\alpha(s)$ has poles at all nontrivial zeros of $\zeta(s)$, we conclude that

$$\zeta(s) \ne 0 \text{ for } \sigma > \frac{1}{2}. \checkmark$$

Remark: So RH is equivalent to "the PNT error term is $\sqrt{x}$ ish". In fact,

$\psi(x) - x \ll_\varepsilon x^{\frac{1}{2}+\varepsilon}$ for all $\varepsilon > 0 \Rightarrow RH \Rightarrow$

$$\psi(x) - x \ll \sqrt{x}\log^2 x.$$

Exercise: Let α-RH denote the assertion that $\zeta(s) \neq 0$ for $\sigma > \alpha$.

Prove that

$$\psi(x) - x \ll_\varepsilon x^{\alpha+\varepsilon} \text{ for every } \varepsilon > 0$$

$$\Rightarrow \alpha\text{-RH} \Rightarrow \psi(x) - x \ll x^\alpha \log^2 x.$$

Recall that

$$N(T) = \#\{\text{nontrivial zeros } \rho = \beta + i\gamma \text{ of } \zeta(s) : 0 < \beta < 1,\ 0 \le \gamma \le T\}$$

or $N(T) = \frac{1}{2}(N(T^-) + N(T^+))$

Also define

$$S(T) = \frac{1}{\pi} \arg \zeta\left(\tfrac{1}{2} + iT\right).$$

Theorem 14.1: For $T > 0$,

$$N(T) = \frac{1}{\pi} \arg \Gamma\left(\frac{1}{4} + \frac{iT}{2}\right) - \frac{T}{2\pi} \log \pi + S(T) + 1.$$

– will prove Tuesday.

Corollary 14.2: For $T \geq 2$,

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + S(T) + \frac{7}{8} + O\left(\frac{1}{T}\right).$$

Proof uses Stirling's formula (Thm C.1).

Exercise: Verify that both sides have jump discontinuities of integral size at ordinates T of zeros of $\zeta(s)$.

We've seen $\arg \zeta(s) \ll \log T$, and so

Corollary 14.3:

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$

Exercise: Deduce from this that

$$N(T+1) - N(T) \ll \log T.$$

(maybe Mean Value Theorem)