Tuesday, March 25

RECALL

- $N(T) = \{$nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$: $0<\beta<1,\ 0 \le \gamma \le T\}$ ← with halfway values at
- $S(T) = \frac{1}{\pi} \arg \zeta(\tfrac{1}{2}+iT)$ ← jump discontinuities

Theorem 14.1: For $T>0$,

$$N(T) = \frac{1}{\pi}\arg\Gamma\left(\frac{1}{4}+\frac{iT}{2}\right) - \frac{T}{2\pi}\log\pi + S(T) + 1.$$

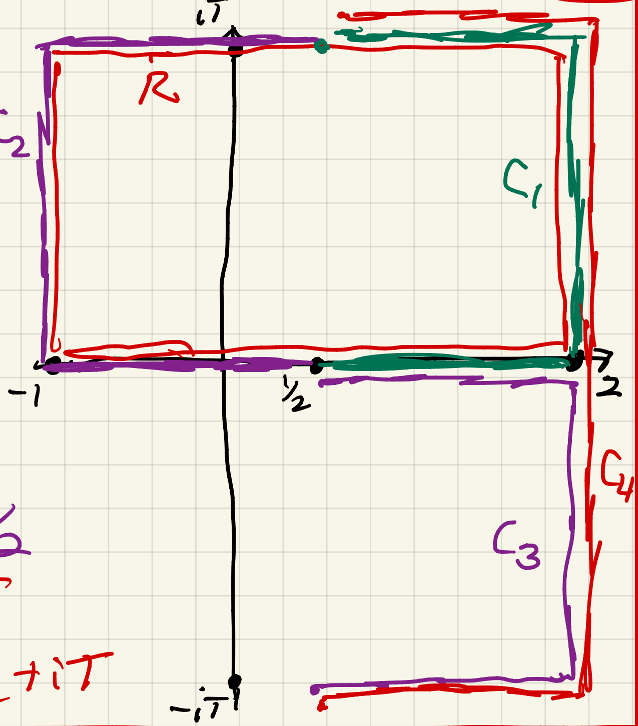
R: corners $-1 \pm iT$ and $2 \pm iT$

C_1: $\frac12$ to 2 to $2+iT$ to $\frac12+iT$

C_2: $\frac12+iT$ to $-1+iT$ to -1 to $\frac12$

C_3: $\frac12-iT$ to $2-iT$ to 2 to $\frac12$

C_4: $\frac12-iT$ to $2-iT$ to $2+iT$ to $\frac12+iT$



Proof of Theorem 14.1: Assume $T>0$ is not the ordinate of a zero of $\xi(s)$. By the argument principle,

$$N(T) = \frac{1}{2\pi i}\oint_R \frac{\xi'}{\xi}(s)\,ds = \frac{1}{2\pi i}\left(\int_{C_1}+\int_{C_2}\right)\frac{\xi'}{\xi}(s)\,ds$$

From the functional equation, $\xi(s) = \xi(1-s)$, we see

$$\int_{C_2}\frac{\xi'}{\xi}(s)\,ds = \int_{C_2} -\frac{\xi'}{\xi}(1-s)\,ds = \int_{C_3}\frac{\xi'}{\xi}(s)\,ds;$$

thus

$$N(T) = \frac{1}{2\pi i}\left(\int_{C_1}+\int_{C_3}\right)\frac{\xi'}{\xi}(s)\,ds = \frac{1}{2\pi i}\int_{C_4}\frac{\xi'}{\xi}(s)\,ds$$

$$= \frac{1}{2\pi i}\log\xi(s)\Big|_{\frac12-iT}^{\frac12+iT}$$ (over the right path)

By Schwarz reflection,

$$N(T) = \frac{1}{2\pi i}\left(\log\xi(\tfrac12+iT) - \log\xi(\tfrac12-iT)\right)$$

$$= \frac{1}{2\pi i}\left(\log\xi(\tfrac12+iT) - \overline{\log\xi(\tfrac12+iT)}\right)$$

$$= \frac{1}{\pi}\operatorname{Im}\log\xi(\tfrac12+iT) = \frac{1}{\pi}\arg\xi(\tfrac12+iT).$$

$$N(T) = \frac{1}{\pi}\arg \xi(\tfrac{1}{2}+iT)$$

$$= \frac{1}{\pi}\Big(\cancel{\arg\tfrac{1}{2}}^{0} + \arg(\tfrac{1}{2}+iT) + \arg(-\tfrac{1}{2}+iT)$$

$$+ \arg \zeta(\tfrac{1}{2}+iT) + \arg\Gamma\Big(\frac{1}{4}+\frac{iT}{2}\Big) - \frac{T}{2}\log\pi\Big)$$

from the definition $\xi(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma(\frac{s}{2})$
$\times \pi^{-s/2}$.

Thus

$$N(T) = S(T) + \frac{1}{\pi}\arg\Gamma\Big(\frac{1}{4}+\frac{iT}{2}\Big) - \frac{T}{2\pi}\log\pi$$

$$+ \underbrace{\frac{1}{\pi}\Big(\arg(\tfrac{1}{2}+iT) + \arg(-\tfrac{1}{2}+iT)\Big)}$$

Exercise: this equals 1. //

"Algorithm" for verifying Riemann hypothesis up to height T.

(1) Find some zeros.

- Code the function $\xi(\frac{1}{2}+it)$.
- Note that FE + reflection $\Rightarrow$

$\xi(1-\bar{s}) = \overline{\xi(s)}$; so $\xi(\frac{1}{2}+it) \in \mathbb{R}$.

- Sample $\xi(\frac{1}{2}+it)$ at many ordinates t and look for sign changes. Each sign change guarantees a zero of $\xi(s)$.
- In this way, for example, we can locate 29 zeros of $\xi(s)$ up to height 100.

(2) Count the zeros

- Code the function $S(T) = \frac{1}{\pi}\arg\zeta(\frac{1}{2}+iT)$.
- Derive an explicit version of Corollary 14.2:

$$\Big|N(T) - \Big(\frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{7}{8} + S(T)\Big)\Big| \le \frac{C}{T}.$$

Exercise!

- Calculate $N(T)$ using this inequality (note: $N(T)$ is an integer!).
- In this way we can calculate $N(100) = 29$.

We just verified RH up to height 100. (and all 29 zeros are simple)

Next topic: primes in arithmetic progressions (primes $p \equiv a \pmod q$)

— note: $(a,q)=1$ is necessary

Problem: • $\sum_{\substack{n \in \mathbb{N} \\ n \equiv a \pmod q}} n^{-s}$ doesn't have an Euler product

• $\prod_{\substack{p \\ p \equiv a \pmod q}} (1-p^{-s})^{-1}$ doesn't have meromorphic continuation/functional equation

Dirichlet's idea: do something more algebraic

Characters of finite abelian groups

Definition: A character χ of a group G is a group homomorphism $\chi: G \to \mathbb{C}^\times$.

Example: $G = \mathbb{Z}/4\mathbb{Z}$.

- $\chi(0)=1$, $\chi(1)=i$, $\chi(2)=-1$, $\chi(3)=-i$.
- $\chi_0(0)=\chi_0(1)=\chi_0(2)=\chi_0(3)=1$.

Note: χ_0 — the trivial character — exists for any G.

Let $\hat{G} = \{\text{all characters } \chi: G \to \mathbb{C}^\times\}$.

Then $\hat{G}$ is a group under pointwise multiplication.

$\hat{G} = \{\text{all characters } \chi : G \to \mathbb{C}^\times\}$

We'll prove the following for all finite abelian groups G:

- $\hat{G} \cong G$
- For any fixed $\chi \in \hat{G}$,

$$(*) \quad \sum_{g \in G} \chi(g) = \begin{cases} \#G, & \text{if } \chi = \chi_0, \\ 0, & \text{if } \chi \neq \chi_0. \end{cases}$$

- For any fixed $g \in G$,

$$(**) \quad \sum_{\chi \in \hat{G}} \chi(g) = \begin{cases} \#G, & \text{if } g = e, \\ 0, & \text{if } g \neq e. \end{cases}$$

Step 1: G is cyclic, $G \cong \mathbb{Z}/n\mathbb{Z}$.

Since $1 \in G$ has order n, $\chi(1)$ must be an nth root of unity; then $\chi(k) = \chi(1)^k$ is determined. So the n characters are

$$\chi_j(k) = e^{2\pi i jk/n} \quad (0 \leq j \leq n-1).$$

Just check $(*)$ and $(**)$ by hand.

Step 2: If $G \cong G_1 \times G_2$, then the three statements would follow from $\hat{G} \cong \hat{G}_1 \times \hat{G}_2$; true because:

- if $\chi_1 \in \hat{G}_1$ and $\chi_2 \in \hat{G}_2$ then $(\chi_1, \chi_2) \in \hat{G}$;
- if $\chi \in \hat{G}$ then define $\chi_1(g_1) = \chi((g_1, e))$ and $\chi_2(g_2) = \chi((e, g_2))$. Then $\chi_j \in \hat{G}_j$ and $\chi = (\chi_1, \chi_2) = \chi_1\chi_2$.

Step 3: Every finite abelian group is the direct product of (several) cyclic groups:

- primary (or elementary) decomposition: $G \cong \mathbb{Z}/q_1\mathbb{Z} \times \cdots \times \mathbb{Z}/q_k\mathbb{Z}$ where each q_i is a prime power.
- invariant factor decomposition: $G \cong \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_\ell\mathbb{Z}$ where $d_1 | d_2 | \cdots | d_\ell$.

Most important to us is the case
$$G = (\mathbb{Z}/q\mathbb{Z})^\times \text{ for some } q \in \mathbb{N}.$$

Given $\chi_G \in \widehat{(\mathbb{Z}/q\mathbb{Z})^\times}$, there is a closely related function $\chi : \mathbb{Z} \to \mathbb{C}$ defined by
$$\chi(n) = \begin{cases} \chi_G(n + q\mathbb{Z}), & \text{if } (n,q) = 1, \\ 0, & \text{if } (n,q) > 1. \end{cases}$$

These χ are called <u>Dirichlet characters</u> modulo q.

<u>Exercise:</u> Equivalently, a Dirichlet character $(\bmod\ q)$ is a completely multiplicative function $\chi : \mathbb{Z} \to \mathbb{C}$ with period q, whose support is exactly $\{n \in \mathbb{Z} : (n,q) = 1\}$.

The group of Dirichlet characters $(\bmod\ q)$ is isomorphic to $(\mathbb{Z}/q\mathbb{Z})^\times$, so has $\phi(q)$ elements.

<u>Example:</u> The first row is a Dirichlet character $(\bmod\ 5)$; the second row is a Dirichlet character $(\bmod\ 6)$.

...	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	...
	0	1	i	$-i$	-1	0	1	i	$-i$	-1	0	1	i	$-i$	-1	0	1	i	
	1	0	0	0	-1	0	1	0	0	0	-1	0	1	0	0	0	-1	0	

<u>Example:</u> For every $q \in \mathbb{Z}$, there is a <u>principal</u> (trivial) character
$$\chi_0(n) = \begin{cases} 1, & \text{if } (n,q) = 1, \\ 0, & \text{if } (n,q) > 1. \end{cases}$$

This is the identity element in the group of Dirichlet characters.

<u>Example:</u> If q is prime, the Legendre symbol $\chi(a) = \left(\frac{a}{q}\right)$ is a Dirichlet character.

Remarks:

- Euler's theorem $a^{\phi(q)} \equiv 1 \pmod{q}$ when $(a,q)=1$ implies that all nonzero values of Dirichlet characters $\pmod{q}$ must be $\phi(q)^{\text{th}}$ roots of unity: if $(a,q)=1$, then

$$\chi(a)^{\phi(q)} \underset{\uparrow}{=} \chi(a^{\phi(q)}) \underset{\uparrow}{=} \chi(1) \underset{\uparrow}{=} 1$$

completely multiplicative — periodic — any group homomorphism

- Note that $\chi(q-1) = \chi(-1) = \pm 1$ since $(-1)^2 = 1$.

Orthogonality relations (as an exer): (Corollary 4.5)

- If $\chi \pmod{q}$ is a Dirichlet character, then

$$\sum_{n=0}^{q-1} \chi(n) = \begin{cases} \phi(q), & \text{if } \chi = \chi_0, \\ 0, & \text{if } \chi \neq \chi_0. \end{cases}$$

- If $n \in \mathbb{Z}$ is fixed, then

$$\sum_{\chi \pmod{q}} \chi(n) = \begin{cases} \phi(q), & \text{if } n \equiv 1 \pmod{q}, \\ 0, & \text{if } n \not\equiv 1 \pmod{q}. \end{cases}$$