Thursday, March 27

Group Work #9 next Tuesday

RECALL

A **Dirichlet character** (mod q) is a completely multiplicative function $\chi : \mathbb{Z} \to \mathbb{C}$ with period q, whose support is $\{n : (n,q)=1\}$.

Example: For every $q \in \mathbb{Z}$, there is a **principal (trivial) character**

$$\chi_0(n) = \begin{cases} 1, & \text{if } (n,q)=1, \\ 0, & \text{if } (n,q)>1. \end{cases}$$

- If $\chi \pmod q$ is fixed, then

$$(*) \quad \sum_{n=0}^{q-1} \chi(n) = \begin{cases} \phi(q), & \text{if } \chi=\chi_0, \\ 0, & \text{if } \chi \neq \chi_0. \end{cases}$$

- If $n \in \mathbb{Z}$ is fixed, then

$$(**) \quad \sum_{\chi \pmod q} \chi(n) = \begin{cases} \phi(q), & \text{if } n \equiv 1 \pmod q, \\ 0, & \text{if } n \not\equiv 1 \pmod q. \end{cases}$$

"Orthogonality relations"

Definition: Given $\chi \pmod q$, define the associated **Dirichlet L-function**

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}.$$

Let's figure out the abscissas of (absolute) convergence for $L(s,\chi)$.

- $\displaystyle\sum_{n=1}^{\infty} |\chi(n) n^{-s}| \leq \sum_{n=1}^{\infty} 1 \cdot n^{-\sigma}$ converges when $\sigma > 1$; so $\sigma_a \leq 1$.

Exercise: Show that the series defining $L(1,\chi)$ does not converge absolutely, and that $L(1,\chi_0)$ does not converge at all. (Hint: look only at $n \equiv 1 \pmod q$.)

From the exercise, we conclude

- $\sigma_a = 1$ for all χ
- $\sigma_c = \sigma_a = 1$ for χ_0 (any q).

Since χ is completely multiplicative, we have the Euler product

$$L(s,\chi) = \prod_p \left(1 + \frac{\chi(p)}{p^s} + \frac{\chi(p^2)}{p^{2s}} + \cdots\right)$$
$$= \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

for $\sigma > 1$.

Exercise: If χ_0 is the principal character (mod q), show that when $\sigma > 1$,

$$L(s,\chi_0) = \zeta(s) \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right).$$

Conclude that $L(s,\chi_0)$ can be continued meromorphically to all of $\mathbb{C}$, with its only pole being a simple pole at $s=1$ with residue $\phi(q)/q$.

Exercise: Show that

$$\sum_{n=1}^{\infty} \chi(n)\Lambda(n) n^{-s} = -\frac{L'}{L}(s,\chi).$$

for $\sigma > 0$.

Note that $(*)$ + periodicity means that if χ is nonprincipal, then $\sum_{m<n\leq m+q} \chi(n) = 0$ for any $m \in \mathbb{Z}$. Consequently, given $x \in \mathbb{N}$, write $x = qv + w$ for $v \in \mathbb{Z}$ and $0 \leq w < q$; then if $A_\chi(x) = \sum_{n\leq x} \chi(n)$, we have

$$A_\chi(x) = \sum_{n\leq w} \chi(n) + \sum_{k=1}^{v} \sum_{(k-1)q<n\leq kq} \chi(n) = \sum_{n\leq w} \chi(n);$$

in particular,

$$|A_\chi(x)| \leq \sum_{n\leq w} |\chi(n)| \leq w < q.$$

Exercise: get the better bound $|A_\chi(x)| \leq \frac{\phi(q)}{2}$.

By Theorem 1.3,

$$\sigma_c = \limsup_{x\to\infty} \frac{\log |A_\chi(x)|}{\log x} \leq \limsup_{x\to\infty} \frac{\log q}{\log x} = 0.$$

Exercise: Verify $\sigma_c \geq 0$. (e.g., look at $s=0$)

Next goal: nonvanishing of $L(1,\chi)$.

Standard proofs of $L(1,\chi) \neq 0$ divide Dirichlet characters into three categories:

- χ is principal (nonvanishing is trivial because of pole)
- χ is quadratic, meaning $\chi^2 = \chi_0$ (equivalently, χ takes only real values)

(The real characters are principal or quadratic)

- χ is complex, meaning $\chi^2 \neq \chi_0$ (χ takes at least one nonreal value).

Here is a sketch of a proof that if χ is complex, then $L(1+it,\chi) \neq 0$ for any $t \in \mathbb{R}$, and in particular $L(1,\chi) \neq 0$.

We recall the inequality

$$|\zeta(\sigma)^3 \zeta(\sigma+it)^4 \zeta(\sigma+2it)| \geq 1$$

for $\sigma > 1$.

Exercise: Prove the analogous inequality

$$|L(\sigma,\chi_0)^3 L(\sigma+it,\chi)^4 L(\sigma+2it,\chi^2)| \geq 1.$$

If $L(1+it,\chi) = 0$, then the function

$$L(s,\chi_0)^3 L(s+it,\chi)^4 L(s+2it,\chi^2), \text{ at } s=1,$$

has (triple pole) $\times$ (at least a quadruple zero) $\times$ (analytic near $s=1$) ← true if $t \neq 0$ or χ is complex.

and thus would have a zero at $s=1$, contradicting the exercise inequality (just to the right of $s=1$).

This argument proves:

- $L(1+it,\chi) \neq 0$ for any χ and any $t \in \mathbb{R} \setminus \{0\}$;
- $L(1,\chi) \neq 0$ for any complex χ.

Proof that $L(1,\chi)\neq 0$ for quadratic χ:

Define $r = \chi * 1$, so $r(n) = \sum_{d\mid n} \chi(d)$.

- <u>Claim 1</u>: $r(n) \geq 0$, and $r(m^2) \geq 1$.

<u>Proof</u>: r is multiplicative, so it suffices to check prime powers.

$$r(p^k) = \chi(1)+\chi(p)+\chi(p)^2+\cdots+\chi(p)^k.$$

If $\chi(p)=\begin{cases}1,\\0,\\-1\end{cases}$ then $r(p^k)=\begin{cases}k+1\\1\\(1 \text{ if } k \text{ is } \cancel{\text{odd}}\ \text{even}),\ (0 \text{ if } k \text{ is } \cancel{\text{even}}\ \text{odd}).\end{cases}$

- <u>Fact</u>: $R(s) = \sum_{n=1}^{\infty} r(n) n^{-s} = L(s,\chi)\zeta(s)$.

- If $L(1,\chi)=0$, then $R(s)$ would be analytic for all $\sigma>0$.

By Landau's theorem (Chapter 1), applicable since $r(n)\geq 0$, $\sum_{n=1}^{\infty} r(n) n^{-s}$ must converge for $\sigma>0$.

- On the other hand, at $s=1/2$,

$$\sum_{n=1}^{\infty} r(n) n^{-1/2} \geq \sum_{m=1}^{\infty} r(m^2)(m^2)^{-1/2}$$

$$\geq \sum_{m=1}^{\infty} 1\cdot m^{-1} = \infty,$$

so the series diverges at $s=1/2$. ✗.

<u>Back to arithmetic progressions</u>: Extend (**) as follows: if $(a,q)=1$

$$\frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \overline{\chi(a)}\,\chi(n) = \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a^{-1})\chi(n)$$

$$= \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a^{-1}n)$$

$$= \begin{cases}1, & \text{if } a^{-1}n \equiv 1 \pmod q\\ 0, & \text{otherwise}\end{cases}$$

$$= \begin{cases}1, & \text{if } n \equiv a \pmod q,\\ 0, & \text{if } n \not\equiv a \pmod q.\end{cases}$$

Consequently, for $\sigma > 1$,

$$\sum_{\substack{n\in\mathbb{N} \\ n\equiv a \pmod q}} n^{-s} = \sum_{n\in\mathbb{N}} n^{-s} \frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi(a)}\, \chi(n)$$

$$= \frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi(a)} \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

$$= \frac{1}{\phi(q)} \sum_{\chi \pmod q} L(s,\chi)\, \overline{\chi(a)}$$

and

$$\sum_{\substack{n\in\mathbb{N} \\ n\equiv a \pmod q}} \Lambda(n) n^{-s} = \cdots = \frac{1}{\phi(q)} \sum_{\chi \pmod q} \left(-\frac{L'}{L}(s,\chi)\right) \overline{\chi(a)}.$$

Let's write this as

$$\sum_{\substack{n\in\mathbb{N} \\ n\equiv a \pmod q}} \Lambda(n) n^{-s} = -\frac{1}{\phi(q)} \frac{L'}{L}(s,\chi_0) - \frac{1}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi\neq\chi_0}} \overline{\chi(a)}\, \frac{L'}{L}(s,\chi).$$

The sum is analytic near $s=1$ (because nonprincipal $L(s,\chi)$ have no poles and don't vanish at $s=1$).

Thus the RHS has a (simple) pole at $s=1$ (of residue $1/\phi(q)$).

In particular,

$$\lim_{\sigma\to 1^+} \sum_{n\equiv a \pmod q} \frac{\Lambda(n)}{n^{\sigma}} = \infty$$

$$\sum_{n\equiv a \pmod q} \frac{\Lambda(n)}{n^{1}} \quad \text{by Monotone Convergence.}$$

Also,

$$\sum_{\substack{n=p^k \\ k\geq 2}} \frac{\Lambda(n)}{n} = \sum_p \sum_{k=2}^{\infty} \frac{\log p}{p^k} = \sum_p \frac{\log p}{p(p-1)} < \infty.$$

So

$$\sum_{p\equiv a \pmod q} \frac{\log p}{p} = \infty.$$

INFINITELY MANY $p \equiv a \pmod q$!