Tuesday, March 4
Group Work #6 today

RECALL

- Theorem 6.7: Let $c>0$ be such that $\zeta(s)\neq 0$ for $\sigma \geq 1-\frac{c}{\log\tau}$. Uniformly for $\sigma \geq 1-\frac{c/2}{\log\tau}$: $\frac{\zeta'}{\zeta}(s) \ll \log\tau$ $\quad \tau=|t|+4$

- $$\frac{1}{2\pi i}\oint_R -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds = x$$

where R is the rectangle with corners $\sigma_0 \pm iT$ and $\sigma_1 \pm iT$.

$$\begin{cases} x>T>e \\ \sigma_0 = 1+1/\log x \\ \sigma_1 = 1-\frac{c/2}{\log(T+4)} \end{cases}$$

- $$\cancel{\psi_0(x)}\ \psi(x) = \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds + O\left(\frac{x}{T}\log^2 x\right)$$

Estimating the top edge of R:

$x^{\sigma_0} = ex.$

$$\leq \int_{\sigma_0+iT}^{\sigma_1+iT} \left|-\frac{\zeta'}{\zeta}(s)\right|\frac{x^{\sigma_0}}{|s|}\,|ds|$$

$$\ll \cancel{(\sigma_0-\sigma_1)}\cdot\cancel{\log(T+4)}\,\frac{x^{\sigma_0}}{T} \ll \frac{x}{T}.$$

- bottom edge is the same

Estimating the left edge of R:

$$\leq \int_{\sigma_1-iT}^{\sigma_1+iT} \log\tau\cdot\frac{x^{\sigma_1}}{|s|}\,|ds|$$

but not quite, near $s=1$

$$\leq 2\int_1^T \log T\,\frac{x^{\sigma_1}}{t}\,dt + \int_{-1}^{1}\frac{1}{1-\sigma_1}\frac{x^{\sigma_1}}{\sigma_1}\,dt$$

$$\ll x^{\sigma_1}\left(\log^2 T+\log T\right) \ll x^{\sigma_1}\log^2 x.$$

Conclusion:

$$\psi(x) = x + O\left(\frac{x}{T}\log^2 x + x^{\sigma_1}\log^2 x\right).$$

Next we pick the best value for T...

$\psi(x) = x + O\left(\frac{x}{T}\log^2 x + x^{\sigma_1}\log^2 x\right)$.

Error term is $x\log^2 x\left(\frac{1}{T} + x^{-\frac{c/2}{\log(T+4)}}\right)$

To minimize, choose T such that ($x > T > e$ say) $\frac{1}{T} = x^{-\frac{c/2}{\log T}}$.

$$-\log T = -\frac{c/2}{\log T}\cdot\log x$$

$$\log^2 T = \frac{c}{2}\log x$$

$$T = \exp\left(\sqrt{\tfrac{c}{2}\log x}\right).$$

(does satisfy $x > T > e$ when x is large enough)

With this choice of T,

$$\psi(x) = x + O\left(x\log^2 x\cdot\exp\left(-\sqrt{\tfrac{c}{2}\log x}\right)\right)$$
$$= x + O\left(x\exp\left(-c'\sqrt{\log x}\right)\right)$$ for any $0 < c' < \sqrt{c/2}$.

Prime Number Theorem.

$$\psi(x) = x + O\left(x e^{-c\sqrt{\log x}}\right)$$

for some $c > 0$.

From Group Work #1:

- better than $\psi(x) = x + O_A\left(\frac{x}{\log^A x}\right)$ for any $A > 0$)
- not as good as $\psi(x) = x + O_\delta(x^{1-\delta})$ for any $\delta > 0$.

Recall $\theta(x) = \sum_{p\le x}\log p$ and $\pi(x) = \#\{p \le x\}$.

From Group Work #3:

- $\theta(x) = \psi(x) + O(\sqrt{x}) = x + O\left(x e^{-c\sqrt{\log x}}\right)$
- $\pi(x) = \frac{\theta(x)}{\log x} + O\left(\frac{\theta(x)}{\log^2 x}\right)$

$$= \frac{x}{\log x} + \cancel{O\left(\frac{x}{\log x}e^{-c\sqrt{\log x}}\right)} + \underline{O\left(\frac{x}{\log^2 x}\right)}.$$

CAN DO BETTER (today's Group Work)