Thursday, March 6

We have proved the prime number theorem!
There exists some $c>0$ such that

- $\psi(x) = \sum_{n\le x} \Lambda(n) = x + O\left(xe^{-c\sqrt{\log x}}\right)$
- $\theta(x) = \sum_{p\le x} \log p = x + O\left(xe^{-c\sqrt{\log x}}\right)$
- $\pi(x) = \sum_{p\le x} 1 = li(x) + O\left(xe^{-c\sqrt{\log x}}\right)$

where $li(x) = \int_2^x \frac{1}{\log t}\,dt \sim \frac{x}{\log x}$.

General technique: • Perron's formula

$$\sum_{n\le x} a_n = \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0+iT}\left(\sum_{n=1}^{\infty} a_n n^{-s}\right)\frac{x^s}{s}\,ds + \text{errors.}$$

- pull contour to the left
- pole(s) of integrand → main terms
- contour estimates → error terms

Example: $\sum_{n\le x}\mu(n) = \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\frac{1}{\zeta(s)}\frac{x^s}{s}\,ds$ + errors

No poles inside the zero-free region!
So main term is 0.

Exercise: $\sum_{n\le x}\mu(n) \ll xe^{-c\sqrt{\log x}}$.

Remarks:

- In fact $\sum_{n\le x}\mu(n) = o(x)$ is equivalent to the prime number theorem
- For $b\in\{-1,0,1\}$, let $M_b(x) = \#\{n\le x : \mu(n)=b\}$

Note that
$M_1(x) + M_{-1}(x) = \#\{n\le x : n \text{ is squarefree}\} \sim \frac{6}{\pi^2}x$.

Also,
$M_1(x) - M_{-1}(x) = \sum_{n\le x}\mu(n) = o(x)$.

It follows that
$M_1(x) \sim \frac{3}{\pi^2}x$, $M_{-1}(x) \sim \frac{3}{\pi^2}x$.

So PNT $\Leftrightarrow$ $\mu(n)=1$ and $\mu(n)=-1$ are asymptotically equally likely.

Similarly for $\lambda(n) = (-1)^{\Omega(n)}$ which is 1 if n has an even number of prime factors counted with multiplicity and -1 if ... odd ... . $\sum_{n=1}^{\infty} \lambda(n) n^{-s} = \frac{\zeta(2s)}{\zeta(s)}$.

We get $\sum_{n\le x} \lambda(n) \ll x e^{-c\sqrt{\log x}}$.

Example (main terms only): $d(n) = \#\{\text{divisors of } n\}$

$$\sum_{n=1}^{\infty} d(n) n^{-s} = \zeta(s)^2, \text{ so}$$

$$\sum_{n\le x} d(n) \approx \frac{1}{2\pi i}\int \zeta(s)^2 \frac{x^s}{s}\,ds.$$

Main term: residue of $\zeta(s)^2 \frac{x^s}{s}$ at $s=1$.

Let's work out the Laurent expansion of $\zeta(s)^2 \frac{x^s}{s}$ around $s=1$:

- $\zeta(s) = \frac{1}{s-1} + C_0 + C_1(s-1) + \cdots$ ← Euler's constant

- $x^s = x e^{(s-1)\log x} = x\left(1 + (s-1)\log x + \frac{1}{2}(s-1)^2 \log^2 x + \cdots\right)$

- $\frac{1}{s} = \frac{1}{1+(s-1)} = 1 - (s-1) + (s-1)^2 - \cdots$

Thus $\zeta(s)^2 \frac{x^s}{s} =$

$$\left(\frac{1}{s-1} + C_0 + C_1(s-1)\cdots\right)^2 \left(1 + (s-1)\log x + (s-1)^2 \log^2 x \cdots\right) x$$
$$\times \left(1 - (s-1) + (s-1)^2 \cdots\right)$$

$$= \left(\frac{1}{(s-1)^2} + \frac{2C_0}{s-1} + 2C_1 + C_0 + \cdots\right)$$
$$\times\ x\left(1 + (\log x - 1)(s-1) + \cdots (s-1)^2 \cdots\right)$$

$$= \frac{x}{(s-1)^2} + \frac{1}{s-1}\, x(\log x - 1 + 2C_0) + \cdots$$

So the residue is $x\log x + (2C_0 - 1)x$.

Examples: $a_n = \frac{1}{n}$.

- $\sum_{n=1}^{\infty} \frac{1}{n} n^{-s} = \zeta(s+1)$, so

$$\sum_{n\le x} \frac{1}{n} \hat{=} \frac{1}{2\pi i}\int \zeta(s+1)\frac{x^s}{s}\,ds.$$

Rightmost singularity is a pole at $s=0$.

- $\zeta(s+1) = \frac{1}{s} + C_0 + \cdots$
- $\frac{1}{s} = \frac{1}{s}$
- $x^s = 1 + s\log x + s^2\log^2 x + \cdots$

So $\zeta(s+1)x^s\frac{1}{s} = \frac{1}{s}\left(\frac{1}{s}+C_0+\cdots\right)(1+s\log x+\cdots)$

$$= \frac{1}{s^2} + \frac{1}{s}(\log x + C_0) + \cdots$$

has residue $\log x + C_0$ at $s=0$.

Example: $a_n = \frac{\mu(n)}{n}$.

- $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} n^{-s} = \frac{1}{\zeta(s+1)}$.

So $\sum_{n\le x}\frac{\mu(n)}{n} \hat{=} \frac{1}{2\pi i}\int \frac{1}{\zeta(s+1)}\frac{x^s}{s}\,ds$ ...

residue at $s=0$ equals 0 since $\frac{1}{\zeta(s+1)}$ has a zero at $s=0$.

Perron method: $\sum_{n\le x}\frac{\mu(n)}{n} \ll e^{-c\sqrt{\log x}}$.

In particular,

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n} = 0.$$

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{0}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{0}{8} + \cdots$$

$$= 0.$$

Next topic: look at further analytic continuation of $\zeta(s)$.

We need some helper functions from complex analysis.

The Gamma function (Appendix C in MV)

Three definitions:

(E) Euler: For $\sigma > 0$,

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\,dx.$$

(G) Gauss: For $s \in \mathbb{C} \setminus \{0,-1,-2,-3,\ldots\}$,

$$\Gamma(s) = \lim_{N\to\infty} \frac{N^s N!}{s(s+1)(s+2)\cdots(s+N)}.$$

(W) Weierstrass: For $s \in \mathbb{C} \setminus \{0,-1,-2,-3,\ldots\}$,

$$\Gamma(s) = \frac{e^{-C_0 s}}{s} \prod_{n=1}^{\infty} \frac{e^{s/n}}{1+s/n}.$$

... don't look very equivalent ...

(W) $\Rightarrow$ (G): take N^{th} partial product and use $\sum_{n\le x} 1/n = \log x + C_0 + O(\frac{1}{x})$.

(G) $\Rightarrow$ (E): repeated integration by parts gives

$$\frac{N^s N!}{s(s+1)\cdots(s+N)} = N^s \int_0^1 (1-y)^N y^{s-1}\,dy$$

$$= \int_0^N \left(1-\frac{x}{N}\right)^N x^{s-1}\,dx,$$

take limits using dominated convergence theorem.

Properties: • $\Gamma(1) = 1$. Trivial from (E) or (G); possible from (W).

• Functional equation $\Gamma(s+1) = s\Gamma(s)$.

- easy from (G); integrate (E) by parts (using $\Gamma(s) = \frac{\Gamma(s+1)}{s}$ extends the definition (E) to $\sigma > -1$, $\sigma > -2$, ..., except at $s = 0, -1, -2, \ldots$)

• Consequence: $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$.

($\Gamma(1) = 0! = 1$)

- $\Gamma(s) \neq 0$, so $1/\Gamma(s)$ is entire.
  (easy from (W) and convergence of infinite products)
- $\Gamma(s)$ has simple poles at $s = 0, -1, -2, \dots$ and the residue of $\Gamma(s)$ at $s = -n$ $(n \geq 0)$ is $\frac{(-1)^n}{n!}$. (use functional equation)

So $\frac{1}{\Gamma(s)} = 0$ for $n = 0, -1, -2, \dots$.

Hence $\frac{1}{\Gamma(s)\Gamma(1-s)} = 0$ for all $n \in \mathbb{Z}$.

Reflection formula (equation (C.6)):

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \qquad (*)$$

Proof uses another formula of Weierstrass:

$$\sin z = z \prod_{k=1}^{\infty}\left(1 - \left(\frac{z}{k\pi}\right)^2\right).$$

Proof of $(*)$ from (G):

$$\Gamma(s)\Gamma(1-s) = \lim_{N\to\infty} \frac{N^s N!}{s(s+1)\cdots(s+N)} \cdot \frac{N^{1-s} N!}{(1-s)(2-s)\cdots(N+1-s)}$$

$$= \lim_{N\to\infty} \frac{N}{s(N+1-s)} \prod_{k=1}^{N} \frac{k^2}{(s+k)(k-s)}$$

$$= \pi \lim_{N\to\infty} \frac{N}{\pi s(N+1-s)} \prod_{k=1}^{N} \frac{1}{1-\left(\frac{s}{k}\right)^2}$$

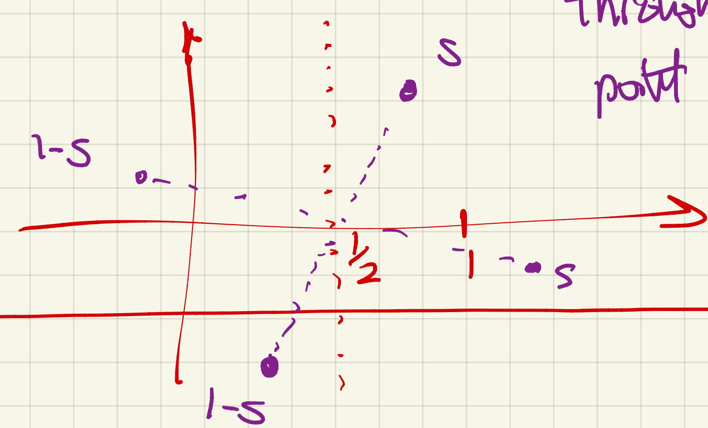
$$= \pi \underbrace{\frac{1}{\pi s} \prod_{k=1}^{\infty} \frac{1}{1-\left(\frac{s}{k}\right)^2}} = \pi \cdot \frac{1}{\sin(\pi s)}. \quad //$$

Note $(*)$ relates the values of Γ at

s and $1-s$

$\sigma + it$ $1 - \sigma - it$

(reflection through the point $s = \frac{1}{2}$).



• Duplication formula (Legendre)

(equation (C.9)):

$$\Gamma(s)\Gamma(s+\tfrac{1}{2}) = \sqrt{\pi}\cdot 2^{1-2s}\,\Gamma(2s).$$

– Exercise from (G).

Remark on (W) definition

If we wanted to build an infinite product with poles at $s=-1,-2,-3,\dots$, we might try $\prod_{k=1}^{\infty}\frac{1}{1+s/k}$. Problem: this product doesn't converge for $s\neq 0$, (converges $\Leftrightarrow \sum_{k=1}^{\infty}\frac{s}{k}$ converges, but that's the harmonic series)

But the $e^{s/k}$ in $\prod_{k=1}^{\infty}\frac{e^{s/k}}{1+s/k}$ is a helpful "convergence factor":

$$\frac{e^{z}}{1+z} = 1+O(|z|^2) \text{ near } z=0.$$