Math 539—Group Work #1

Thursday, January 9, 2025

1. *Let h and k be functions with $h(x) > 2$ and $k(x) > 2$ for all sufficiently large x.*

(a) *Prove that $h(x) \ll k(x)$ implies $\log h(x) \ll \log k(x)$.*

(b) *Show, via a counterexample, that the converse to part (a) is false.*

(a) Let $C \geq 1$ be a constant such that $h(x) \leq Ck(x)$. Then $\log h(x) \leq \log k(x) + \log C < \log k(x) + \frac{\log k(x)}{\log 2} \log C$ for x sufficiently large, by the lower bound on $k(x)$. In other words, $\log h(x) < D \log k(x)$ with $D = 1 + \frac{\log C}{\log 2}$ for x sufficiently large; this means that $\log h(x) \ll \log k(x)$ for x sufficiently large.

Note that some lower bound like $k(x) > 2$ is necessary, as the counterexample $h(x) = 3$, $k(x) = e^{1/x}$ shows.

(b) A simple counterexample is $h(x) = e^{3x}$ and $k(x) = e^x$. Then $\log h(x) = 3x \ll x = \log k(x)$, but it is certainly not true that $e^{3x} \ll e^x$ (their quotient is e^{2x} which is not bounded). Indeed, if $k(x)$ is any function tending to infinity, then choosing $h(x) = k(x)^\alpha$ for some constant $\alpha > 1$ creates such a counterexample.

2. *Again let h and k be functions with $h(x) > 2$ and $k(x) > 2$ for all sufficiently large x. Prove that $\log h(x) = o\big(\log k(x)\big)$ implies $h(x) = o\big(k(x)\big)$. Conclude that $\log h(x) = o\big(\log k(x)\big)$ implies $h(x) \ll k(x)$.*

We are given that $\lim_{x\to\infty} \frac{\log h(x)}{\log k(x)} = 0$; in particular, we may choose X so large that $\frac{\log h(x)}{\log k(x)} \leq \frac{1}{2}$ for all $x \geq X$. This means that $h(x) \leq k(x)^{1/2}$ and therefore $\frac{h(x)}{k(x)} \leq \frac{1}{\sqrt{k(x)}}$ for $x \geq X$. On the other hand, note that $0 \leq \lim_{x\to\infty} \frac{\log 2}{\log k(x)} < \lim_{x\to\infty} \frac{\log h(x)}{\log k(x)} = 0$ and therefore $\lim_{x\to\infty} \frac{\log 2}{\log k(x)} = 0$, which means that $\lim_{x\to\infty} \log k(x) = \infty$ and therefore $\lim_{x\to\infty} k(x) = \infty$ as well. Therefore, $0 \leq \lim_{x\to\infty} \frac{h(x)}{k(x)} \leq \lim_{x\to\infty} \frac{1}{\sqrt{k(x)}} = 0$, which shows that $\lim_{x\to\infty} \frac{h(x)}{k(x)} = 0$ and therefore that $h(x) = o\big(k(x)\big)$.

Knowing that $h(x) = o\big(k(x)\big)$, we may choose Y so large that $\frac{h(x)}{k(x)} \leq 1$ for all $x \geq Y$, and therefore $h(x) \leq k(x)$ for all $x \geq Y$. In particular, $h(x) \ll k(x)$ for sufficiently large x. (And, if we had more carefully specified the domain on which our functions are defined and included some continuity hypothesis, then we could trivially deduce that $h(x) \ll k(x)$ for all x in the functions' domain.)

In my opinion, this implication from Question 2 is the single most important tool to use when trying to compare error terms in analytic number theory.

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3. *For all real numbers* $A > 0$ *and* $0 < b < 1$ *and* $\varepsilon > 0$, *show that*

$$\log^A x \ll \exp(\log^b x) \ll x^\varepsilon$$

uniformly for $x \geq 1$, *where the implicit constants may depend upon A, b, and* ε*—just not on* x.

(Remark: the "uniformly" in x *is already implied by the definition of* $\ll$, *so it means exactly the same thing to say just that the estimates hold "for* $x \geq 1$*"; but sometimes people add the "uniformly" for emphasis.)*

The key is to note that by question #2, it suffices to show that $\log(\log^A x) = o\big(\log(\exp(\log^b x))\big)$ and that $\log(\exp(\log^b x)) = o\big(\log(x^\varepsilon)\big)$, or in other words that $A \log\log x = o\big(\log^b x\big)$ and $\log^b x = o\big(\varepsilon \log x\big)$.

For the first assertion, we consider the limit of their quotient $\lim_{x\to\infty} \frac{A\log\log x}{\log^b x}$. This is an "$\frac{\infty}{\infty}$" indeterminate form, so we may apply l'Hôpital's rule. (We do remember to check that limits are indeterminate before invoking l'Hôpital's rule, right?) We obtain

$$\lim_{x\to\infty} \frac{A\log\log x}{\log^b x} = \lim_{x\to\infty} \frac{A/(x\log x)}{b(\log x)^{b-1}/x} = \frac{A}{b} \lim_{x\to\infty} \frac{1}{(\log x)^b} = 0,$$

since $b > 0$. The second assertion is even simpler, since the limit of their quotient is simply

$$\lim_{x\to\infty} \frac{\log^b x}{\varepsilon \log x} = \frac{1}{\varepsilon} \lim_{x\to\infty} \frac{1}{(\log x)^{1-b}} = 0$$

since $b < 1$.

Remark: this problem isn't just for torturing students … functions of the type $\exp(\log^b x)$ really do come up in analytic number theory, and it's good to remember that the way to figure out how big they are (in practice) is to just take logarithms until the comparison is easy. For example, if you see $\exp(\sqrt{\log x})$ in the wild, you'll be able to know that that's a function that's "bigger than any power of $\log x$ but smaller than any power of x", as this problem showed.

4. *Suppose that* $f(x)$ *and* $g(x)$ *are differentiable functions on* $[1, \infty)$, *with* g *increasing, and that* $f(x) \ll g(x)$. *Is it true that* $f'(x) \ll g'(x)$*?*

Very much not! Take for example $f(x) = x + \sin(x^2)$ and $g(x) = x + 1$, so that $f'(x) = 1 + 2x\cos(x^2)$ and $g'(x) = 1$. This isn't something weird about the $\ll$ relation, as the same example demonstrates that the $\leq$ relation has the same property. Morally, there's no reason such an implication should be true—a bound on a function says nothing about a bound on its derivative—so we just have to avoid letting the symbols lead us into a pitfall. (For example, if we know that $f(x) = x^2 + O(x)$, we *cannot* conclude that $f'(x) = 2x + O(1)$.)

5. *Can you find functions* $f(x)$ *and* $g(x)$ *such that for all real numbers* $A > 0$ *and* $0 < b < 1$ *and* $\varepsilon > 0$,

$$\log^A x \ll f(x) \ll \exp(\log^b x) \ll g(x) \ll x^\varepsilon$$

uniformly for $x \geq 1$*? (Again the implicit constants may depend upon A, b, and* ε*—just not on* x.*)*

Valid choices include $f(x) = \exp\big((\log\log x)^C\big)$ for all $C > 1$, and $g(x) = \exp\big((\log x)/(\log\log x)^d\big)$ for all $d > 0$. (The implicit constants will depend on C and d with these choices.) There are all kinds of functions we can fit in the cracks between the cracks and so on—for example, where does $\exp(\exp(\sqrt{\log\log x}))$ go?