Math 539—Group Work #2
Tuesday, January 21, 2025

Group work criteria: Start from the top and understand one problem fully before moving on to the next one; quality is more important than quantity (although these group work problems are designed so that ideally you will be able to finish them all). I will be going from group to group during the hour, paying attention to the following aspects.

1. Effective communication—including both listening and speaking, with respect for other people and their ideas
2. Engagement with, and curiosity about, the material (for instance, how far might something generalize?)
3. Boldness—suggesting ideas, and trying plans even when they're incomplete
4. Obtaining valid solutions (which are understood by everyone in the group) to the given problems

Definition: For any positive integer k, the *generalized divisor function* $d_k(n)$ is the number of ordered k-tuples $(m_1, \dots, m_k)$ of positive integers such that $m_1 \times \cdots \times m_k = n$. For example, $d_1(n) = 1$ for all $n \ge 1$, while $d_2(n) = d(n)$.

1. Prove that $d_j * d_k = d_{j+k}$ for all positive integers j and k.

Given positive integers n and ℓ, define the set

$$T_\ell(n) = \bigl\{(m_1, \dots, m_\ell) \in \mathbb{N}^\ell \colon m_1 \times \cdots \times m_\ell = n\bigr\}$$

(so that $\#T_\ell(n) = d_\ell(n)$), and define a union of Cartesian products of sets

$$U_{j,k}(n) = \bigcup_{a \mid n} \bigl(T_j(a) \times T_k(\tfrac{n}{a})\bigr).$$

Then it suffices to show that

$$\#T_{j+k}(n) = d_{j+k}(n) = (d_j * d_k)(n) = \sum_{a \mid n} d_j(a) d_k(\tfrac{n}{a}) = \#U_{j,k}(n).$$

But there is an obvious function from $U_{j,k}(n)$ to $T_{j+k}(n)$: for any $a \mid n$, send the element $\bigl((m_1, \dots, m_j), (q_1, \dots, q_k)\bigr)$ of $T_j(a) \times T_k(\frac{n}{a})$ to $(m_1, \dots, m_j, q_1, \dots, q_k)$. One can check that this function is well-defined and invertible: its inverse sends the element $(m_1, \dots, m_{j+k})$ of $T_{j+k}(n)$ to $\bigl((m_1, \dots, m_j), (m_{j+1}, \dots, m_{j+k})\bigr)$, which is an element of $T_j(a) \times T_k(\frac{n}{a})$ with $a = m_1 \times \cdots \times m_j$. This bijection proves that $\#T_{j+k}(n) = \#U_{j,k}(n)$ as desired.

One could also prove the identity $d_j * 1 = d_j * d_1 = d_{j+1}$ for all positive integers j (by a similar but perhaps simpler counting argument) and then deduce, using the associativity of Dirichlet convolution and induction, that $d_j * d_k = \underbrace{1 * \cdots * 1}_{j+k} = d_{j+k}$.

Note that this induction method also makes it easy to see that d_k is multiplicative for all integers $k \ge 1$, since the Dirichlet convolution of two multiplicative functions is automatically multiplicative, and $d_1 = 1$ is certainly multiplicative.

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2. *Given the identity proved in question #1:*

   (a) *What do you think a sensible way to define $d_{1/2}$ would be?*
   (b) *Calculate $d_{1/2}(539)$ and $d_{1/2}(16)$, given your "sensible" definition above.*

   (a) We definitely want $d_{1/2}$ to satisfy $d_{1/2} * d_{1/2} = d_1 = 1$, that is,

$$\sum_{a|n} d_{1/2}(a) d_{1/2}(\tfrac{n}{a}) = 1 \tag{1}$$

   for all $n \in \mathbb{N}$. After thinking about part (b) for a bit, we realize that it's also sensible to ask that $d_{1/2}(n) > 0$ for all n. (Indeed, in the best possible world, we would like $d_{1/2}$ to be a multiplicative function, in harmony with the fact that d_k is multiplicative for all $k \in \mathbb{N}$ as remarked in #1 above; but we will not need to assume that $d_{1/2}$ has this property.)

   (b) First, the desired identity (1) for $n = 1$ becomes simply $d_{1/2}(1)^2 = 1$, so that $d_{1/2}(1) = \pm 1$. (Indeed, if $d_{1/2}$ is a function satisfying the identity (1), then so is the function $-d_{1/2}$; so it makes sense that we would have a sign choice like this at some point.) Being sensible people, we choose $d_{1/2} = 1$ and proceed. For prime powers p^r for small values of r, the identity (1) becomes

$$\begin{aligned}
1 &= d_{1/2}(1)d_{1/2}(p) + d_{1/2}(p)d_{1/2}(1) \\
1 &= d_{1/2}(1)d_{1/2}(p^2) + d_{1/2}(p)d_{1/2}(p) + d_{1/2}(p^2)d_{1/2}(1) \\
1 &= d_{1/2}(1)d_{1/2}(p^3) + d_{1/2}(p)d_{1/2}(p^2) + d_{1/2}(p^2)d_{1/2}(p) + d_{1/2}(p^3)d_{1/2}(1) \\
1 &= d_{1/2}(1)d_{1/2}(p^4) + d_{1/2}(p)d_{1/2}(p^3) \\
&\qquad + d_{1/2}(p^2)d_{1/2}(p^2) + d_{1/2}(p^3)d_{1/2}(p) + d_{1/2}(p^4)d_{1/2}(1),
\end{aligned}$$

   which we can solve recursively, obtaining $d_{1/2}(p) = \frac{1}{2}$, $d_{1/2}(p^2) = \frac{3}{8}$, $d_{1/2}(p^3) = \frac{5}{16}$, and $d_{1/2}(p^4) = \frac{35}{128}$; in particular, $d_{1/2}(16) = \frac{35}{128}$.

   A similar computation when $n = pq$ and $n = p^2q$ for distinct primes p and q yields

$$\begin{aligned}
1 &= d_{1/2}(1)d_{1/2}(pq) + d_{1/2}(p)d_{1/2}(q) + d_{1/2}(q)d_{1/2}(p) + d_{1/2}(pq)d_{1/2}(1) \\
1 &= d_{1/2}(1)d_{1/2}(p^2q) + d_{1/2}(p)d_{1/2}(pq) + d_{1/2}(p^2)d_{1/2}(q) \\
&\qquad + d_{1/2}(q)d_{1/2}(p^2) + d_{1/2}(pq)d_{1/2}(p) + d_{1/2}(p^2q)d_{1/2}(1),
\end{aligned}$$

   which we can solve recursively to obtain $d_{1/2}(pq) = \frac{1}{4}$ and $d_{1/2}(p^2q) = \frac{3}{16}$; in particular, $d_{1/2}(539) = \frac{3}{16}$. Note that the values we have computed do satisfy $d_{1/2}(pq) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = d_{1/2}(p)d_{1/2}(q)$ and $d_{1/2}(p^2q) = \frac{3}{16} = \frac{3}{8} \cdot \frac{1}{2} = d_{1/2}(p^2)d_{1/2}(q)$, giving evidence that perhaps $d_{1/2}$ is a multiplicative function (a fact which we will more fully justify in #5 below).

   [We remark that it is possible to prove the following general statement: if $f * f = g$ where g is multiplicative and $f(1) = 1$, then f is multiplicative. The proof essentially proceeds by showing that given g, there is only one such function f, and then showing that the multiplicative function generated by the values of f on prime powers satisfies the correct identity. Indeed, a similar approach works for $f * f * \cdots * f = g$. For a different approach when all functions are real-valued, see D. Rearick, *Operators on algebras of arithmetic functions*, Duke Math. J. **35** (1968), 761–766.]

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3.

(a) *For all positive integers k, prove that $\sum_{n=1}^{\infty} d_k(n)n^{-s}$ converges, in a suitable half-plane, to $\zeta(s)^k$.*

(b) *For which real numbers α is it true that $\sum_{n\le x} d_k(n) \ll_{k,\alpha} x^\alpha$?*

(a) We proceed by induction on k, the case $k = 1$ being obvious (in the half-plane $\sigma > 1$) since $d_1 = 1$ identically. Assuming that $\sum_{n=1}^{\infty} d_k(n)n^{-s} = \zeta(s)^k$ for $\sigma > 1$: note that in particular $\sum_{n=1}^{\infty} |d_k(n)n^{-s}| = \sum_{n=1}^{\infty} d_k(n)n^{-\sigma} < \infty$, so that the series converges absolutely for $\sigma > 1$, as does the series defining $\zeta(s)$ itself. (In general, any Dirichlet series with nonnegative coefficients satisfies $\sigma_a = \sigma_c$.) Then by Theorem 1.8, the series $\sum_{n=1}^{\infty} d_{k+1}(n)n^{-s} = \sum_{n=1}^{\infty}(d_k * d_1)(n)n^{-s}$ (where we have used problem #1) also converges absolutely for $\sigma > 1$, to

$$\left(\sum_{n=1}^{\infty} d_k(n)n^{-s}\right)\left(\sum_{n=1}^{\infty} d_1(n)n^{-s}\right) = \zeta(s)^k\zeta(s) = \zeta(s)^{k+1}.$$

(b) Applying Theorem 1.3 with $a_n = d_k(n)$ yields

$$\limsup_{x\to\infty} \frac{\log\left(\sum_{n\le x} d_k(n)\right)}{\log x} = \sigma_c = 1.$$

In other words, for every $\varepsilon > 0$, we know that $\log\left(\sum_{n\le x} d_k(n)\right)/\log x < 1+\varepsilon$ for all sufficiently large x, but that $\log\left(\sum_{n\le x} d_k(n)\right)/\log x > 1-\varepsilon$ for a sequence of values of x tending to infinity. We conclude that $\sum_{n\le x} d_k(n) < x^{1+\varepsilon}$ for x sufficiently large in terms of ε (and k), which implies that $\sum_{n\le x} d_k(n) \ll_{k,\alpha} x^\alpha$ for all $\alpha > 1$; and we also conclude that $\sum_{n\le x} d_k(n) \not\ll x^\alpha$ for $\alpha < 1$ (why? there's a slightly nontrivial step). Of course we could also argue that $\sum_{n\le x} d_k(n) \not\ll x^\alpha$ for $\alpha < 1$ simply by noting that $d_k(n) \ge 1$ for all $n \in \mathbb{N}$.

As it turns out, we don't yet have enough information to decide the case of $\alpha = 1$, namely whether $\sum_{n\le x} d_k(n) \ll_k x$ (although we will in the next couple of weeks). If $d_k(n)$ were bounded then the answer would be yes, or even if $d_k(n)$ were occasionally large but "usually" bounded; if $d_k(n) > \log n$ (say) for all n then the answer would be no. It turns out that none of these assertions hold for $d_k(n)$. Note that we are asking how large $d_k(n)$ is "on average", which will turn out to be an easier problem; nevertheless, we will eventually learn pointwise bounds for $d_k(n)$ as well.

4. Given the identity proved in question #3(a):

(a) *What do you think a sensible way to define d_z would be for any complex number z?*

(b) *Calculate $d_i(539)$ and $d_i(16)$, given your "sensible" definition above. (Here, $i = \sqrt{-1}$.)*

(a) It seems that we would love for $\zeta(s)^z$ to be a Dirichlet series, so we can define $d_z(n)$ to be the coefficient of n^{-s} in the Dirichlet series for $\zeta(s)^z$. And indeed, the fact that $\zeta(s)$ has an Euler product gives us great hope, since then

$$\zeta(s)^z = \prod_p \left(1 - p^{-s}\right)^{-z} \tag{2}$$

seems like a sensible enough function.

(b) One way to calculate the coefficients of the Dirichlet series hidden in the right-hand side of equation (2) is to write

$$\zeta(s)^z = \prod_p \exp\big(z \log(1-p^{-s})^{-1}\big) = \prod_p \exp\big(z(p^{-s} + \tfrac12 p^{-2s} + \tfrac13 p^{-3s} + \cdots)\big).$$

For example,

$$\begin{aligned}
\zeta(s)^i &= \prod_p \exp\big(i(p^{-s} + \tfrac12 p^{-2s} + \tfrac13 p^{-3s} + \tfrac14 p^{-4s} + \cdots)\big) \\
&= \prod_p \sum_{k=0}^\infty \frac{1}{k!} i^k (p^{-s} + \tfrac12 p^{-2s} + \tfrac13 p^{-3s} + \tfrac14 p^{-4s} + \cdots)^k \\
&= \prod_p \bigg(1 + i(p^{-s} + \tfrac12 p^{-2s} + \tfrac13 p^{-3s} + \tfrac14 p^{-4s} + \cdots) - \frac12 (p^{-s} + \tfrac12 p^{-2s} + \tfrac13 p^{-3s} + \cdots)^2 \\
&\qquad - \frac{i}{6}(p^{-s} + \tfrac12 p^{-2s} + \cdots)^3 + \frac{1}{24}(p^{-s} + \cdots)^4 + \cdots \bigg),
\end{aligned}$$

where all of the terms required to compute the coefficients up to p^{-4s} have been included explicitly. Simplifying,

$$\zeta(s)^i = \prod_p \left(1 + ip^{-s} + \left(-\frac12 + \frac{i}{2}\right)p^{-2s} + \left(-\frac12 + \frac{i}{6}\right)p^{-3s} - \frac{5}{12}p^{-4s} + \cdots\right).$$

Therefore, if we write $\zeta(s)^i = \sum_{n=1}^\infty d_i(n) n^{-s}$, then $d_i(n)$ is multiplicative (given the Euler product above) and, on prime powers, takes the values $d_i(p) = i$, $d_i(p^2) = -\frac12 + \frac{i}{2}$, $d_i(p^3) = -\frac12 + \frac{i}{6}$, $d_i(p^4) = -\frac{5}{12}, \dots$. In particular, $d_i(539) = d_i(7^2)d_i(11) = (-\frac12 + \frac{i}{2})i = -\frac12 - \frac{i}{2}$ and $d_i(16) = d_i(2^4) = -\frac{5}{12}$.

5. Can you write down a formula for $d_{1/2}(p^r)$ as a function of r?

We could find the key to this problem either through experimentation as above, or through having seen before the Maclaurin series representation for arbitrary powers of $1+x$, namely $(1+x)^w = \sum_{k=0}^\infty \binom{w}{k} x^k$ where

$$\binom{w}{k} = \frac{w(w-1)(w-2)\cdots(w-k+1)}{k!}$$

is the generalized binomial coefficient (defined for nonnegative integers k but for all complex numbers w). Either way, we are led to observe that equation (2) can be written as

$$\zeta(s)^z = \prod_p (1-p^{-s})^{-z} = \prod_p \sum_{k=0}^\infty \binom{-z}{k}(-p^{-s})^k,$$

and so the function $d_z(n)$ can be defined as the multiplicative function whose value on prime powers p^r equals $(-1)^r\binom{-z}{r}$. (Check that this is the familiar answer when $z = 2$!) In particular, $d_{1/2}(n)$ is the multiplicative function whose value on prime powers p^r equals

$$(-1)^r \binom{-1/2}{r} = (-1)^r \frac{-\frac12(-\frac12 - 1)(-\frac12 - 2)\cdots(-\frac12 - r + 1)}{r!} = \frac{(2r)!}{4^r (r!)^2}$$

after some simplification.