Math 539—Group Work #4

Tuesday, February 4, 2025

1. *The goal of this problem is to find the average value of $n/\phi(n)$. (Note that there's no reason the answer should be the reciprocal of the average value of $\phi(n)/n$.)*

(a) *Find an explicit formula for the arithmetic function h that has the property that*

$$\sum_{d|n} h(d) = \frac{n}{\phi(n)}$$

for all positive integers n.

(b) *For the function h from part (a), prove that*

$$\sum_{d\le x} \frac{h(d)}{d} = \prod_p \left(1 + \frac{1}{p(p-1)}\right) + O_\varepsilon(x^{-1+\varepsilon})$$

for every $\varepsilon > 0$. (You may use the following fact: $\phi(n) \gg_\varepsilon n^{1-\varepsilon}$ for every $\varepsilon > 0$.)

(c) *Prove that the average value of $n/\phi(n)$ is $\zeta(2)\zeta(3)/\zeta(6)$.*

(a) The function $n/\phi(n)$ is multiplicative (being the quotient of two multiplicative functions), and so we know that the function h will also be multiplicative (since, by Möbius inversion, $h(n) = \mu(n) * \frac{n}{\phi(n)}$ is the convolution of two multiplicative functions). Therefore it suffices to calculate $h(p^r)$ for prime powers p^r. From the Möbius inversion formula,

$$\begin{aligned} h(p^r) &= \sum_{d|p^r} \mu(d) \frac{p^r/d}{\phi(p^r/d)} \\ &= \mu(1)\frac{p^r}{\phi(p^r)} + \mu(p)\frac{p^{r-1}}{\phi(p^{r-1})} + \mu(p^2)\frac{p^{r-2}}{\phi(p^{r-2})} + \cdots + \mu(p^r)\frac{1}{\phi(1)} \\ &= \frac{p^r}{\phi(p^r)} - \frac{p^{r-1}}{\phi(p^{r-1})} + 0 + \cdots + 0 = \begin{cases} 1/(p-1), & \text{if } r = 1, \\ 0, & \text{if } r \ge 2. \end{cases} \end{aligned}$$

In other words,

$$h(n) = \left.\begin{cases} \prod_{p|n} \frac{1}{p-1}, & \text{if } n \text{ is squarefree,} \\ 0, & \text{otherwise} \end{cases}\right\} = \frac{\mu^2(n)}{\phi(n)}.$$

(b) Suspecting that the left-hand side actually converges as $x \to \infty$, we look at the tail of the series: for any $0 < \varepsilon < 1$,

$$0 \le \sum_{d>x} \frac{h(d)}{d} = \sum_{d>x} \frac{\mu^2(d)/\phi(d)}{d} < \sum_{d>x} \frac{1}{d\phi(d)} \ll_\varepsilon \sum_{d>x} \frac{1}{d^{2-\varepsilon}} \ll_\varepsilon \frac{1}{x^{1-\varepsilon}}.$$

In particular, the tail tends to 0 as $x \to \infty$, and therefore the infinite series $\sum_{d=1}^\infty \frac{h(d)}{d}$ converges; since all its terms are nonnegative, it converges absolutely, and therefore (since $\frac{h(d)}{d}$ is multiplicative) we can write it as its Euler product

$$\sum_{d=1}^\infty \frac{h(d)}{d} = \prod_p \left(1 + \frac{h(p)}{p} + \frac{h(p^2)}{p^2} + \cdots\right) = \prod_p \left(1 + \frac{1}{p(p-1)} + 0 + \cdots\right).$$

Putting the pieces together, we see that indeed

$$\sum_{d\le x}\frac{h(d)}{d}=\sum_{d=1}^{\infty}\frac{h(d)}{d}+O\bigg(\sum_{d>x}\frac{h(d)}{d}\bigg)=\prod_p\left(1+\frac{1}{p(p-1)}\right)+O_\varepsilon(x^{-1+\varepsilon}).$$

(c) By our convolution method,

$$\frac{1}{x}\sum_{n\le x}\frac{n}{\phi(n)}=\frac{1}{x}\sum_{n\le x}\sum_{d\mid n}h(d)=\frac{1}{x}\sum_{d\le x}h(d)\left\lfloor\frac{x}{d}\right\rfloor=\sum_{d\le x}\frac{h(d)}{d}+O\bigg(\frac{1}{x}\sum_{d\le x}h(d)\bigg).$$

This error term is

$$\frac{1}{x}\sum_{d\le x}h(d)=\frac{1}{x}\sum_{d\le x}\frac{\mu^2(d)}{\phi(d)}\ll_\varepsilon\frac{1}{x}\sum_{d\le x}\frac{1}{x^{1-\varepsilon}}\ll_\varepsilon\frac{1}{x}x^\varepsilon=o(1),$$

while the main term, by part (b), is

$$\sum_{d\le x}\frac{h(d)}{d}=\prod_p\left(1+\frac{1}{p(p-1)}\right)+o(1);$$

therefore the average value of $n/\phi(n)$ is the infinite product above. Finally, the factor in that product can be rewritten as

$$\frac{p^2-p+1}{p(p-1)}=\frac{(p^2-p+1)(p+1)}{p(p-1)(p+1)}=\frac{p^3+1}{p(p^2-1)}=\frac{(p^3+1)(p^3-1)}{p(p^2-1)(p^3-1)}=\frac{p^6-1}{p(p^2-1)(p^3-1)},$$

and so the average value in question is

$$\begin{aligned}\prod_p\left(1+\frac{1}{p(p-1)}\right)&=\prod_p\frac{p^6-1}{p(p^2-1)(p^3-1)}\\&=\prod_p\frac{1-p^{-6}}{(1-p^{-2})(1-p^{-3})}\\&=\bigg(\prod_p(1-p^{-6})^{-1}\bigg)^{-1}\prod_p(1-p^{-2})^{-1}\prod_p(1-p^{-3})^{-1}=\frac{\zeta(2)\zeta(3)}{\zeta(6)}.\end{aligned}$$

(For the record, the average value of $\phi(n)/n$ is $6/\pi^2\approx0.607927$, while $\zeta(2)\zeta(3)/\zeta(6)\approx1.94360$ is greater than $\pi^2/6\approx1.64493$.)

Side comment: we saw in class that $\zeta(2)=\pi^2/6$, and it turns out that $\zeta(6)=\pi^6/945$. (Later this semester you'll learn how to prove these identities.) But the number $\zeta(3)$ is more mysterious. It wasn't until 1978 that Apéry proved that $\zeta(3)$ is irrational (and thus $\zeta(3)$ is sometimes called Apéry's constant); and while we don't expect $\zeta(3)/\pi^3$ to be rational, I think that's still an open problem.

(continued on next page)

2.

(a) *Prove that* $\displaystyle\sum_{m\le x}\sum_{\substack{n\le x\\(m,n)=1}} 1 = \sum_{d\le x}\mu(d)\left\lfloor\frac{x}{d}\right\rfloor^2$. *Hint: what does* $\sum_{d\mid(m,n)}\mu(d)$ *equal?*

(b) *Write down the rigorous definition of what a number theorist refers to as "the probability that two randomly chosen positive integers are relatively prime to each other", and calculate it.*

(c) *A* lattice point (in the plane) *is a point* (x, y) *such that both* x *and* y *are integers. A lattice point is* visible from the origin *if the line segment between it and the origin contains no other lattice points besides the endpoints. What is "the probability that a randomly chosen lattice point in the plane is visible from the origin"? (Note: in the plane, not in the first quadrant.)*

(d) *Generalize part (c) to lattice points in three-dimensional space; in* k*-dimensional space.*

(a) Following the hint, we can write

$$\sum_{m\le x}\sum_{\substack{n\le x\\(m,n)=1}} 1 = \sum_{m\le x}\sum_{n\le x}\sum_{d\mid(m,n)}\mu(d) = \sum_{d\le x}\mu(d)\sum_{\substack{m\le x\\d\mid m}}\sum_{\substack{n\le x\\d\mid n}} 1 = \sum_{d\le x}\mu(d)\left\lfloor\frac{x}{d}\right\rfloor\left\lfloor\frac{x}{d}\right\rfloor$$

as desired.

(b) Presumably we should sample two positive integers m and n independently and uniformly from the integers up to x, calculate the probability that they are coprime as a function of x, and take the limit as x goes to ∞. That finite probability is exactly

$$\begin{aligned}\frac{1}{\lfloor x\rfloor^2}\sum_{m\le x}\sum_{\substack{n\le x\\(m,n)=1}} 1 &= \frac{1}{\lfloor x\rfloor^2}\sum_{d\le x}\mu(d)\left\lfloor\frac{x}{d}\right\rfloor^2 \\ &= \frac{1}{\lfloor x\rfloor^2}\sum_{d\le x}\mu(d)\left(\frac{x}{d}+O(1)\right)^2 \\ &= \frac{1}{\lfloor x\rfloor^2}\left(x^2\sum_{d\le x}\frac{\mu(d)}{d^2}+O\left(x\sum_{d\le x}\frac{|\mu(d)|}{d}+\sum_{d\le x}|\mu(d)|\right)\right) \\ &= \frac{x^2}{\lfloor x\rfloor^2}\left(\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}+O\left(\sum_{d>x}\frac{\mu(d)}{d^2}\right)\right)+O\left(\frac{x}{\lfloor x\rfloor^2}\sum_{d\le x}\frac{|\mu(d)|}{d}+\frac{1}{\lfloor x\rfloor^2}\sum_{d\le x}|\mu(d)|\right).\end{aligned}$$

Using $\mu(d)\ll 1$, and $\lfloor x\rfloor \gg x$ for $x\ge 1$ (confirm!), this probability becomes

$$\begin{aligned}\frac{x^2}{\lfloor x\rfloor^2}\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}+O\left(\sum_{d>x}\frac{1}{d^2}+\frac{1}{x}\sum_{d\le x}\frac{1}{d}+\frac{1}{x^2}\sum_{d\le x}1\right) &= \frac{x^2}{\lfloor x\rfloor^2}\frac{1}{\zeta(2)}+O\left(\frac{1}{x}+\frac{1}{x}\log x+\frac{1}{x^2}x\right) \\ &= \frac{x^2}{\lfloor x\rfloor^2}\frac{6}{\pi^2}+O\left(\frac{\log x}{x}\right).\end{aligned}$$

The limit of this expression as $x\to\infty$ is $6/\pi^2$.

(Along the way we saw that the difference between $\lfloor x\rfloor$ and x was insignificant in this calculation, since $x\to\infty$; therefore in practice we usually start such calculations with $1/x$ or $1/x^2$ instead of $1/\lfloor x\rfloor$ or $1/\lfloor x\rfloor^2$.)

(c) We will use the fact that the lattice point (m,n) is visible from the origin if and only if $\gcd(m,n)=1$ (confirm!). After some reflection, we choose to sample lattice points uniformly from the square with vertices $(\pm x, \pm x)$, which contains $(2\lfloor x \rfloor + 1)^2$ lattice points. We therefore want to calculate

$$\frac{1}{(2x+1)^2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ |m|,|n|\le x \\ \gcd(m,n)=1}} 1 = \frac{1}{(2x+1)^2}\Bigg(4 \sum_{1\le m\le x} \sum_{\substack{1\le n\le x \\ \gcd(m,n)=1}} 1 + 4\Bigg),$$

since the counts for the four quadrants are identical (greatest common divisors ignore signs) and there are precisely 4 lattice points on the two axies that are visible from the origin. Using part (b) it is easy to check that the limit of this expression as $x\to\infty$ equals $6/\pi^2$.

(The regions from which these lattice points are sampled can be thought of as a fixed shape, namely the square with vertices $(\pm 1, \pm 1)$, which is then dilated by a factor of x. One can start with other fixed shapes instead and dilate them in the same way; under some conditions—certainly using a convex neighborhood of the origin is sufficient, although that can be loosened quite a bit—the proportion of lattice points that are visible from the origin will still tend to $6/\pi^2$. Research has been done on the quality of the error term in these asymptotic formulas; you can see a paper some colleagues and I wrote for some results in this vein, along with a few pointers to the more fundamental results.)

(d) The method of part (a) generalizes quickly to

$$\sum_{\substack{n_1,\dots,n_k\le x \\ (n_1,\dots,n_k)=1}} 1 = \sum_{n_1,\dots,n_k\le x} \sum_{d\mid(n_1,\dots,n_k)} \mu(d) = \sum_{d\le x}\mu(d) \sum_{\substack{n_1\le x \\ d\mid n_1}} \cdots \sum_{\substack{n_k\le x \\ d\mid n_k}} 1 = \sum_{d\le x}\mu(d)\left\lfloor \frac{x}{d}\right\rfloor^k.$$

The method of part (b) then gives

$$\frac{1}{x^k}\sum_{d\le x}\mu(d)\left\lfloor\frac{x}{d}\right\rfloor^k = \sum_{d\le x}\frac{\mu(d)}{d^k} + O\bigg(\frac{1}{x}\sum_{d\le x}\frac{|\mu(d)|}{d^{k-1}}\bigg) = \frac{1}{\zeta(k)} + O\Big(\frac{1}{x}\Big)$$

(the error term valid when $k\ge 3$). Therefore the “probability” that a random lattice point in $\mathbb{Z}^k$ is visible from the origin turns out to be $1/\zeta(k)$.