Math 539—Group Work #6

Tuesday, March 4, 2025

Define the "logarithmic integral" function $\mathrm{li}(x) = \displaystyle\int_2^x \frac{du}{\log u}$.

1. In this problem, we will explore various ways to write the error term in the prime number theorem for $\pi(x)$.

(a) *Using integration by parts, or otherwise, show that* $\mathrm{li}(x) = \dfrac{x}{\log x} + \displaystyle\int_2^x \frac{du}{\log^2 u} - \frac{2}{\log 2}$.

(b) *Show that* $\mathrm{li}(x) = \dfrac{x}{\log x} + \dfrac{x}{\log^2 x} + \dfrac{2x}{\log^3 x} + O\Big(\dfrac{x}{\log^4 x}\Big)$.

(c) *For any positive integer K, prove that* $\pi(x) = \displaystyle\sum_{k=1}^{K} \frac{(k-1)!x}{\log^k x} + O_K\Big(\frac{x}{(\log x)^{K+1}}\Big)$. *You may assume equation* (1) *below to accomplish this task.*

(d) *For any fixed $\alpha > 2$, deduce that it is not the case that* $\pi(x) = \dfrac{x}{\log x} + O\Big(\dfrac{x}{\log^\alpha x}\Big)$.

(a) Integration by parts (integrating 1 and differentiating $1/\log u$) yields

$$\mathrm{li}(x) = \frac{u}{\log u}\bigg|_2^x - \int_2^x u\Big(-\frac{1}{u\log^2 u}\Big)\,du = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{du}{\log^2 u}.$$

(b) We continue integrating by parts:

$$\begin{aligned}
\mathrm{li}(x) &= \frac{x}{\log x} + \int_2^x \frac{du}{\log^2 u} + O(1) \\
&= \frac{x}{\log x} + \frac{u}{\log^2 u}\bigg|_2^x - \int_2^x u\Big(-\frac{2}{u\log^3 u}\Big)\,du + O(1) \\
&= \frac{x}{\log x} + \frac{x}{\log^2 x} + \int_2^x \frac{2}{\log^3 u}\,du + O(1) \\
&= \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2u}{\log^3 u}\bigg|_2^x - \int_2^x u\Big(-\frac{6}{u\log^4 u}\Big)\,du + O(1) \\
&= \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \int_2^x \frac{6}{\log^4 u}\,du + O(1).
\end{aligned}$$

As for the remaining integral, again we split at some $2 \le y \le x$ and estimate each integral trivially:

$$\int_2^x \frac{6}{\log^4 u}\,du = \int_2^y \frac{6}{\log^4 u}\,du + \int_y^x \frac{6}{\log^4 u}\,du \ll y + x\cdot\frac{1}{\log^4 y},$$

and many choices of y make the right-hand side $\ll x/\log^4 x$ (for example, $y = \sqrt{x}$).

Another way of estimating this last integral: noting that

$$\frac{d}{dx}\Big(\frac{x}{\log^4 x}\Big) = \frac{1}{\log^4 x} - \frac{4}{\log^5 x} \ge \frac{1/2}{\log^4 x} \quad \text{for } \log x \ge 8,$$

we may write (when $x \ge e^8$)

$$\int_2^x \frac{6}{\log^4 u}\,du = \int_2^{e^8} \frac{6}{\log^4 u}\,du + \int_{e^8}^x \frac{6}{\log^4 u}\,du \le \int_2^{e^8} \frac{6}{\log^4 u}\,du + 12\int_{e^8}^x \left(\frac{1}{\log^4 u} - \frac{4}{\log^5 u}\right)du,$$

and therefore

$$\int_2^x \frac{6}{\log^4 u}\,du \ll 1 + \int_{e^8}^x \left(\frac{1}{\log^4 u} - \frac{4}{\log^5 u}\right)du = 1 + \frac{u}{\log^4 u}\bigg|_{e^8}^x \ll \frac{x}{\log^4 x}.$$

(c) Using repeated integration by parts as in part (b), it is easy to prove by induction on K that

$$\mathrm{li}(x) = \sum_{k=1}^K \frac{(k-1)!x}{\log^k x} + \int_2^x \frac{K!}{(\log u)^{K+1}} + O_K(1).$$

(Notice a slight subtlety of the notation: adding K quantities that are each $O(1)$ yields a quantity that is $O_K(1)$, but not necessarily $O(1)$ uniformly in K.) As in part (b), splitting the remaining integral at $y = \sqrt{x}$, say, shows that the integral is $\ll_K x/(\log x)^{K+1}$. Therefore by problem #1(b), there exists an absolute constant $c > 0$ such that

$$\begin{aligned}
\pi(x) &= \mathrm{li}(x) + O\big(x\exp(-c\sqrt{\log x})\big) \\
&= \sum_{k=1}^K \frac{(k-1)!x}{\log^k x} + O_K\bigg(\frac{x}{(\log x)^{K+1}} + x\exp(-c\sqrt{\log x})\bigg) \\
&= \sum_{k=1}^K \frac{(k-1)!x}{\log^k x} + O_K\bigg(\frac{x}{(\log x)^{K+1}}\bigg),
\end{aligned}$$

since $(\log x)^{K+1} \ll_K \exp(c\sqrt{\log x})$ for any K. (No dependence on c is necessary since it is an absolute constant.)

[Note that it is tempting to extend this finite series to an infinite series, writing something like $\mathrm{li}(x) = \sum_{k=1}^\infty \frac{(k-1)!x}{\log^k x}$. However, the ratio test reveals that this series does not converge for any value of x! This is an example of a *divergent series*, where any specific truncation provides a good approximation asymptotically even though the infinite series itself isn't useful.]

(d) Suppose that the estimate did hold; then from part (c) with $K = 2$,

$$\frac{x}{\log x} + O\bigg(\frac{x}{\log^\alpha x}\bigg) = \pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + O\bigg(\frac{x}{\log^3 x}\bigg);$$

after rearranging this becomes

$$\frac{x}{\log^2 x} = O\bigg(\frac{x}{\log^\alpha x} + \frac{x}{\log^3 x}\bigg)$$

which is certainly false when $\alpha > 2$.

2. In this problem, we will give an asymptotic formula for $\pi(x)$ with a better error term than what we saw in class.

(a) *Show that*

$$\pi(x) - \mathrm{li}(x) = \frac{\theta(x) - x}{\log x} + \frac{2}{\log 2} + \int_2^x \frac{\theta(u) - u}{u\log^2 u}\,du.$$

(b) *Suppose that* $c > 0$ *is a constant such that* $\theta(x) = x + O\big(x\exp(-c\sqrt{\log x})\big)$. *Prove that*

$$\pi(x) = \mathrm{li}(x) + O\big(x\exp(-c\sqrt{\log x})\big). \tag{1}$$

(a) We can write $\pi(x) = \sum_{p\le x} 1$ in terms of $\theta(x) = \sum_{p\le x}\log p$ using Riemann–Stieltjes integrals:

$$\begin{aligned}\pi(x) = \int_{2-}^{x}\frac{1}{\log u}\,d\theta(u) &= \int_{2-}^{x}\frac{1}{\log u}\,d(\theta(u)-u) + \int_{2-}^{x}\frac{1}{\log u}\,du\\ &= \int_{2-}^{x}\frac{1}{\log u}\,d(\theta(u)-u) + \mathrm{li}(x) - \mathrm{li}(2-).\end{aligned}$$

Rearranging terms, replacing $\mathrm{li}(2-)$ by $\mathrm{li}(2) = 0$ (due to the implicit limit in that lower endpoint that will soon be taken), and integrating by parts, we obtain

$$\begin{aligned}\pi(x) - \mathrm{li}(x) &= \frac{\theta(u)-u}{\log u}\bigg|_{2-}^{x} - \int_{2-}^{x}(\theta(u)-u)\,d\frac{1}{\log u}\\ &= \frac{\theta(x)-x}{\log x} - \frac{0-2}{\log 2} + \int_{2}^{x}(\theta(u)-u)\frac{1}{u\log^2 u}\,du.\end{aligned}$$

(b) From part (a),

$$\begin{aligned}\pi(x) - \mathrm{li}(x) &\ll \frac{x\exp(-c\sqrt{\log x})}{\log x} + 1 + \int_{2}^{x}\frac{u\exp(-c\sqrt{\log u})}{u\log^2 u}\,du \qquad (2)\\ &\ll x\exp(-c\sqrt{\log x}) + \int_{2}^{y}\frac{\exp(-c\sqrt{\log u})}{\log^2 u}\,du + \int_{y}^{x}\frac{\exp(-c\sqrt{\log u})}{\log^2 u}\,du\end{aligned}$$

for any $2 \le y \le x$. Since the integrand is positive and decreasing for $u \ge 2$, it is also bounded, and so

$$\begin{aligned}\pi(x) - \mathrm{li}(x) &\ll x\exp(-c\sqrt{\log x}) + y + (x-y)\frac{\exp(-c\sqrt{\log y})}{\log^2 y}\\ &\ll x\exp(-c\sqrt{\log x}) + y + x\exp(-c\sqrt{\log y}).\end{aligned}$$

A reasonable choice for y seems to be $y = x\exp(-c\sqrt{\log x})$. With this choice,

$$\log y = \log x - c\sqrt{\log y} = (\log x)\left(1 + O\left(\frac{\sqrt{\log y}}{\log x}\right)\right);$$

since $\sqrt{1+O(\varepsilon)} = 1 + O(\varepsilon)$ by the tangent line for $\sqrt{1+t}$ at $t = 0$,

$$\sqrt{\log y} = \sqrt{\log x}\left(1 + O\left(\frac{\sqrt{\log y}}{\log x}\right)\right) = \sqrt{\log x} + O(1).$$

We conclude that

$$\pi(x) - \mathrm{li}(x) \ll x\exp(-c\sqrt{\log x}) + x\exp\left(-c\big(\sqrt{\log x} + O(1)\big)\right) \ll x\exp(-c\sqrt{\log x}),$$

since $\exp(O(1)) \ll 1$.

Alternatively, we can use the "wishful thinking derivative" method we saw in #1(b): since

$$\frac{d}{dx}\big(x\exp(-c\sqrt{\log x})\big) = \exp(-c\sqrt{\log x}) - \frac{c}{2}\frac{\exp(-c\sqrt{\log x})}{\sqrt{\log x}} \gg x\exp(-c\sqrt{\log x}),$$

we have

$$\begin{aligned}\int_2^x \frac{\exp(-c\sqrt{\log u})}{\log^2 u}\,du &\ll \int_2^x \exp(-c\sqrt{\log u})\,du\\ &\ll \int_2^x \frac{d}{du}\big(u\exp(-c\sqrt{\log u})\big)\,du\\ &= x\exp(-c\sqrt{\log x}) - 2\exp(-c\sqrt{\log 2}) \ll x\exp(-c\sqrt{\log x}),\end{aligned}$$

with which the required estimate follows from equation (2).