Math 539—Group Work #7

Thursday, March 13, 2025

Throughout this group work, $x \ge T > 4$ and $\sigma_0 > 1$ and $0 < \delta < 1$ and $\varepsilon > 0$ are real numbers; and R denotes the rectangle (oriented counterclockwise) with corners at $\sigma_0 - iT$, $\sigma_0 + iT$, $\delta + iT$, and $\delta - iT$.

We also include some facts for your convenience:

- Let $\alpha(s) = \sum_{n=1}^\infty a_n n^{-s}$ have abscissa of absolute convergence σ_a. As long as $\sigma_0 > \sigma_a$,

$$\sum_{n\le x} a_n = \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0+iT} \alpha(s)\frac{x^s}{s}\,ds + O\biggl(\frac{x\log x}{T}\max\Bigl\{|a_n|\colon \frac{x}{2} < n < 2x\Bigr\} + \frac{x^{\sigma_0}}{T}\sum_{n=1}^\infty \frac{|a_n|}{n^{\sigma_0}}\biggr).$$

  (This is a consequence of Perron's formula, as in Corollary 5.3.)
- $\zeta(s) \ll_{\delta,\varepsilon} \tau^{1-\sigma}\log\tau$ uniformly on $\bigl\{s\in\mathbb{C}\colon \delta\le\sigma\le 1+1/\log\tau \text{ and } |s-1|\ge\varepsilon\bigr\}$. (This is a consequence of Theorem 1.11.)
- If a meromorphic function $f(z)$ has a pole of order k at $z=1$, then $f(z)\sim c(z-1)^{-k}$ near $z=1$ for some constant c depending on f. More explicitly, if f has no other poles in $\{z\in\mathbb{C}\colon |z-1|\le R\}$, then $|f(z)| \ll |z-1|^{-k}$ on $\{z\in\mathbb{C}\colon |z-1|\le r\}$ (here $0<r<R$ are constants, and the $\ll$-constant may depend on f, r, and R). (This follows from the general theory of Laurent expansions.)
- $\log d(n) < \dfrac{\log n}{\log\log n}\biggl(\log 2 + O\Bigl(\dfrac{1}{\log\log n}\Bigr)\biggr)$ for $n\ge 3$. (This is Theorem 2.11.)

0. Preliminaries:

(a) *Argue that $d(n) \ll_\varepsilon n^\varepsilon$.*

(b) *Argue that*

$$\frac{1}{2\pi i}\oint_R \zeta(s)^2\frac{x^s}{s}\,ds = x\log x + (2C_0-1)x. \tag{1}$$

(a) The maximal order result above shows that $\log d(n) = o_\varepsilon(\varepsilon\log n)$ for any $\varepsilon>0$. We then know (Group Work #1) that we can exponentiate both sides to obtain $d(n) = o_\varepsilon(n^\varepsilon)$, which in particular implies $d(n)\ll_\varepsilon n^\varepsilon$.

(b) The function x^s is entire, while $s=0$ is not inside the rectangle R. Therefore the only pole of the integrand is the double pole of $\zeta(s)$ at $s=1$. We saw in class that the residue of this pole is $x\log x + (2C_0-1)x$, and so the identity (1) follows from the residue theorem.

1.

(a) *Show that the contribution to the integral* (1) *from the right edge of R is*

$$\sum_{n\le x} d(n) + O_\varepsilon\biggl(\frac{x^{1+\varepsilon}}{T} + \frac{x^{\sigma_0}}{T}\zeta(\sigma_0)^2\biggr).$$

Conclude that if $\sigma_0 = 1+1/\log x$, then this contribution is $\displaystyle\sum_{n\le x} d(n) + O_\varepsilon\biggl(\frac{x^{1+\varepsilon}}{T}\biggr)$.

(b) *Suppose that $T \le \frac{1}{2}\sqrt{x}$. Show that the contribution to the integral* (1) *from the top and bottom edges of R is $\ll_\delta (x/T^2)^{\sigma_0} T \log^2 T$. Conclude that if $\sigma_0 = 1 + 1/\log x$, then this contribution is $\ll_{\delta,\varepsilon} x^{1+\varepsilon}/T$.*

(c) *Show that the contribution to the integral* (1) *from the left edge of R is $\ll_\delta x^\delta T^{2-2\delta} \log^2 T$.*

(d) *Conclude from the above calculations that*

$$\sum_{n\le x} d(n) = x\log x + (2C_0 - 1)x + O_{\delta,\varepsilon}\big(x^{(2-\delta)/(3-2\delta)+\varepsilon}\big).$$

(Hint: ignore the x^ε and $\log^2 T$ when deciding what value to choose for T.) In particular, show that

$$\sum_{n\le x} d(n) = x\log x + (2C_0 - 1)x + O_\eta\big(x^{2/3+\eta}\big).$$

(a) Using Perron's formula as given above with $\alpha(s) = \zeta(s)^2$, so that $a_n = d(n)$, we see that the contribution to the integral (1) from the right edge of R is

$$\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} \zeta(s)^2 \frac{x^s}{s}\,ds = \sum_{n\le x} d(n) + O\bigg(\frac{x\log x}{T}\max\Big\{d(n)\colon \frac{x}{2} < n < 2x\Big\} + \frac{x^{\sigma_0}}{T}\sum_{n=1}^\infty \frac{d(n)}{n^{\sigma_0}}\bigg).$$

The sum in the error term is exactly $\zeta(\sigma_0)^2$; on the other hand, by problem 0(a) above, $\max\big\{d(n)\colon x/2 < n < 2x\big\} \ll_\varepsilon (2x)^\varepsilon \ll_\varepsilon x^\varepsilon$. Since $\log x \ll_\varepsilon x^\varepsilon$ as well, we obtain the first assertion. (Note that a function that is $O_\varepsilon(x^{1+2\varepsilon}/T)$ for every $\varepsilon > 0$ is also $O_\varepsilon(x^{1+\varepsilon}/T)$ for every $\varepsilon > 0$, by replacing ε with $\varepsilon/2$.)

If we take $\sigma_0 = 1 + 1/\log x$, then $x^{\sigma_0} = ex \ll x$, while (because of its double pole at $s = 1$) we have $\zeta(\sigma_0) \ll (\sigma_0 - 1)^{-2} = \log^2 x \ll_\varepsilon x^\varepsilon$. These observations establish the second assertion.

(b) The contribution to the integral (1) from the top edge of R is

$$-\frac{1}{2\pi i}\int_{\delta+iT}^{\sigma_0+iT} \zeta(s)^2 \frac{x^s}{s}\,ds = -\frac{1}{2\pi i}\int_\delta^{\sigma_0} \zeta(\sigma+iT)^2 \frac{x^{\sigma+iT}}{\sigma+iT}\,d\sigma$$
$$\ll \int_\delta^{\sigma_0} |\zeta(\sigma+iT)|^2 \frac{|x^{\sigma+iT}|}{|\sigma+iT|}\,d\sigma = \int_\delta^{\sigma_0} |\zeta(\sigma+iT)|^2 \frac{x^\sigma}{|\sigma+iT|}\,d\sigma.$$

From the reference facts, we see that $\zeta(s) \ll_\delta T^{1-\sigma}\log T$ (we can take $\varepsilon = 4$ to verify $|\sigma + iT - 1| \ge T > 4$, and the difference between τ and T is neglible in this error term since $T > 4$ implies $\tau \le 2T$), and of course $|\sigma + iT| \ge T$; and we can extend the lower endpoint to $-\infty$ by the positivity of the integrand. Therefore the contribution from the top edge is

$$\ll_\delta \int_{-\infty}^{\sigma_0} (T^{1-\sigma}\log T)^2 \frac{x^\sigma}{T}\,d\sigma = T\log^2 T\int_{-\infty}^{\sigma_0}\left(\frac{x}{T^2}\right)^\sigma d\sigma$$
$$= T\log^2 T\,\frac{(x/T^2)^{\sigma_0} - 0}{\log(x/T^2)} \ll (x/T^2)^{\sigma_0} T\log^2 T,$$

since $\log(x/T^2) \ge \log 4$ by assumption. (Note that the vanishing of the lower boundary term is due to the fact that $x/T^2 > 1$, which holds by the same assumption. In this case, it would work just as well to simply take the maximum value of the modulus of the integrand multiplied by the length of the path of integration, the latter of which is $\ll 1$ in our context.)

Since the contribution from the bottom edge is exactly the complex conjugate of the contribution from the top edge (up to a minus sign from traversing the edge in the other direction), the same estimate holds for the bottom edge, and the first assertion is established.

If we take $\sigma_0 = 1 + 1/\log x$, then again $x^{\sigma_0} = ex \ll x$, while $(1/T^2)^{\sigma_0} < (1/T^2)^{-1}$, since $\log T < \log x \ll_\varepsilon x^\varepsilon$, we quickly obtain the second assertion as well.

(c) The contribution to the integral (1) from the left edge of R is

$$-\frac{1}{2\pi i}\int_{\delta-iT}^{\delta+iT} \zeta(s)^2 \frac{x^s}{s}\,ds = -\frac{1}{2\pi i}\int_{-T}^{T} \zeta(\delta+it)^2 \frac{x^{\delta+it}}{\delta+it}\,dt$$
$$\ll \int_{-T}^{T} |\zeta(\delta+it)|^2 \frac{|x^{\delta+it}|}{|\delta+it|}\,dt = x^\delta \int_{-T}^{T} \frac{|\zeta(\delta+it)|^2}{|\delta+it|}\,dt.$$

From the reference facts, we see that $\zeta(s) \ll_\delta \tau^{1-\delta}\log\tau$ (we can take $\varepsilon = 1-\delta$ to verify $|\delta + iT - 1| \ge 1-\delta$, and $\ll_{\delta,\varepsilon}$ becomes just $\ll_\delta$ once we choose ε to be a function of δ), while $|\delta+it| \ge \max\{\delta, |t|\} \gg_\delta \tau$ (you can confirm this by considering separately the cases $|t| \le \delta$ and $|t| \ge \delta$). Therefore the contribution from the left edge is (since $\tau = |t|+4$)

$$\ll_\delta x^\delta \int_{-T}^{T} \frac{(\tau^{1-\delta}\log\tau)^2}{\tau}\,dt = 2x^\delta \int_4^{T+4} t^{1-2\delta}\log^2 t\,dt$$
$$\le 2x^\delta \log^2(T+4)\int_0^{T+4} t^{1-2\delta}\,dt$$
$$= 2x^\delta \log^2(T+4)\frac{(T+4)^{2-2\delta}}{2-2\delta}\,dt \ll_\delta x^\delta T^{2-2\delta}\log^2 T.$$

(d) Putting the results of parts (a)–(c) into the identity (1), we see that we have shown

$$\sum_{n\le x} d(n) = x\log x + (2C_0-1)x + O_{\delta,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T} + x^\delta T^{2-2\delta}\log^2 T\right)$$
$$= x\log x + (2C_0-1)x + O_{\delta,\varepsilon}\left(x^\varepsilon\left(\frac{x}{T} + x^\delta T^{2-2\delta}\right)\right)$$

for $T \le \frac{1}{2}\sqrt{x}$ (since $\log T \le \log x \ll_\varepsilon x^\varepsilon$). We choose T to make $x/T = x^\delta T^{2-2\delta}$; that choice is $T = x^{(1-\delta)/(3-2\delta)}$ (which we verify is at most $x^{1/3}$, which is $\le \frac{1}{2}\sqrt{x}$ when $x \ge 64$). With this choice, the above estimate becomes

$$\sum_{n\le x} d(n) = x\log x + (2C_0-1)x + O_{\delta,\varepsilon}\big(x^{(2-\delta)/(3-2\delta)+\varepsilon}\big).$$

In particular, the exponent $(2-\delta)/(3-2\delta)+\varepsilon$ is decreasing in both δ and ε and approaches $\frac{2}{3}$ as $\delta, \varepsilon \to 0+$; so for any $\eta > 0$, there are choices of δ and ε in terms of η such that $(2-\delta)/(3-2\delta)+\varepsilon < \frac{2}{3}+\eta$. We conclude that

$$\sum_{n\le x} d(n) = x\log x + (2C_0-1)x + O_\eta\big(x^{2/3+\eta}\big).$$

[Note that this asymptotic formula has the same main terms, but a weaker (larger) error term, as the formula we obtained from Dirichlet's hyperbola method in Chapter 2. Although we haven't seen it in class, it is not too hard to improve the reference bound on the zeta function from $\zeta(s) \ll \tau^{1-\sigma}\log\tau$ to $\zeta(s) \ll \tau^{(1-\sigma)/2}\log\tau$; this would result in an asymptotic formula with an error term of $O_\eta(x^{1/2+\eta})$, almost but still not quite as good as the elementary method. It is a strange feature of this subject that for arithmetic functions for which the elementary methods work at all, they sometimes tend to give better error bounds than this contour-integration method. Of course, as you might imagine, the contour-integration method is much more versatile.]

2. Let $k \ge 2$ be an integer, and let $d_k(n)$ be the generalized divisor function, namely the number of ordered k-tuples of positive integers whose product equals n (so that $d_2(n) = d(n)$).

(a) *Show that the residue of the function $\zeta(s)^k x^s/s$ at $s = 1$ is equal to $xP_k(\log x)$, where $P_k(T) \in \mathbb{R}[T]$ is a polynomial of degree $k-1$ with leading coefficient $1/(k-1)!$.*
(b) *Show that $d_k(n) \ll_{k,\varepsilon} n^\varepsilon$ for every $\varepsilon > 0$. (Hint: show first that $d_k(n) \le d(n)^k$.)*
(c) *Prove that for every $\eta > 0$,*

$$\sum_{n\le x} d_k(n) = xP_k(\log x) + O_{\eta,k}\big(x^{k/(k+1)+\varepsilon}\big).$$

Conclude that in particular,

$$\sum_{n\le x} d_k(n) = \frac{x(\log x)^k}{(k-1)!} + O_k\big(x(\log x)^{k-1}\big).$$

(a) Raising the Laurent expansion at $s = 1$,

$$\zeta(s) = \frac{1}{s-1} + C_0 + C_1(s-1) + C_2(s-2) + \cdots,$$

to the kth power yields

$$\zeta(s)^k = \frac{1}{(s-1)^k} + \frac{D_{1-k}}{(s-1)^{k-1}} + \cdots + \frac{D_{-1}}{s-1} + D_0 + D_1(s-1) + \cdots.$$

(Here, the C_j and D_j are some constants whose precise values won't concern us here, although C_0 really is Euler's constant; for consistency we write $D_{-k} = 1$.) As we have seen before, known Taylor series for exponential functions and geometric series give

$$\begin{aligned}\frac{x^s}{s} = xe^{(s-1)\log x}\frac{1}{1-(-(s-1))} &= \left(\sum_{j=0}^\infty x\frac{(s-1)^j(\log x)^j}{j!}\right)\left(\sum_{\ell=0}^\infty (-1)^\ell (s-1)^\ell\right)\\ &= \sum_{m=0}^\infty (s-1)^m\left(\sum_{j=0}^m \frac{x(\log x)^j}{j!}(-1)^{m-j}\right),\end{aligned}$$

which begins $x^s/s = x + (x\log x - x)(s-1) + (\frac12 x\log^2 x - x\log x + x)(s-1)^2 + \cdots$. Note that we may write this series as

$$\frac{x^s}{s} = \sum_{m=0}^\infty (s-1)^m xQ_m(\log x),$$

where $Q_m(T)$ is a polynomial of degree m with leading coefficient $1/m!$. It follows that the coefficient of $1/(s-1)$ in the Laurent expansion of $\zeta(s)^k x^s/s$ equals

$$\sum_{n=0}^{k-1} xQ_n(\log x)D_{-1-n} = x\left(Q_{k-1}(\log x) + \sum_{n=0}^{k-2} Q_n(\log x)D_{-1-n}\right),$$

and the problem follows upon taking $P_k(x) = Q_{k-1}(T) + \sum_{n=0}^{k-2} Q_n(T)D_{-1-n}$ and noting that this sum is a polynomial of degree $k-2$.

(b) The suggested inequality $d_k(n) \le d(n)^k$ follows from the definition of $d_k(n)$ as the number of ordered k-tuples of positive integers whose product equals n, since each element of every such k-tuples is a divisor of n. Therefore for every $\eta > 0$, we have $d_k(n) \le d(n)^k \ll_{\eta,k} (n^\eta)^k$ (note that this constant does depend on k, since it is the kth power of the constant implicit in $d(n) \ll_\eta n^\eta$); choosing $\eta = \varepsilon/k$, we conclude that $d_k(n) \ll_{\varepsilon,k} n^\varepsilon$.

(c) We follow the same outline as in problem 1, starting with the contour integral evaluation

$$\frac{1}{2\pi i} \oint_R \zeta(s)^k \frac{x^s}{s}\, ds = xP_k(\log x) \tag{2}$$

in the notation of part (a). We recognize that $\zeta(s)^k = \sum_{n=1}^\infty d_k(n)n^{-s}$. Using Perron's formula as in problem 1(a), we find that the contribution from the right edge of R is

$$\sum_{n \le x} d_k(n) + O_{\varepsilon,k}\biggl(\frac{x^{1+\varepsilon}}{T} + \frac{x^{\sigma_0}}{T}\zeta(\sigma_0)^k\biggr),$$

where we have used part (b) to bound $d_k(n)$ in the first error term. Choosing $\sigma_0 = 1 + 1/\log x$ results in $\zeta(\sigma_0)^k \ll \log^k x \ll_{\varepsilon,k} x^\varepsilon$, and so the second error term gets absorbed into the first as before.

If we assume that $T \le \frac{1}{2}\sqrt[k]{x}$, then the same calculation as in problem 1(b) shows that the contribution to the integral (2) from the top and bottom edges of R is

$$\ll_{\delta,k} \int_{-\infty}^{\sigma_0} (T^{1-\sigma}\log T)^k \frac{x^\sigma}{T}\, d\sigma \ll (x/T^k)^{\sigma_0} T^{k-1}\log^k T,$$

which again is $\ll_{\delta,\varepsilon,k} x^{1+\varepsilon}/T$ when we choose $\sigma_0 = 1 + 1/\log x$. Similarly, the same calculation as in problem 1(c) shows that the contribution to the integral (2) from the left edge of R is

$$\ll_{\delta,k} x^\delta \int_{-T}^{T} \frac{(\tau^{1-\delta}\log\tau)^k}{\tau}\, dt \ll_{\delta,k} x^\delta T^{k-k\delta}\log^k T.$$

Combining these estimates (still assuming $T \le \frac{1}{2}\sqrt[k]{x}$ and thus using $\log^k T \ll_{\varepsilon,k} x^\varepsilon$) yields the asymptotic formula

$$\sum_{n \le x} d_k(n) = xP_k(\log x) + O_{\delta,\varepsilon,k}\biggl(x^\varepsilon\biggl(\frac{x}{T} + x^\delta T^{k-k\delta}\biggr)\biggr).$$

The optimal choice (which makes the two error terms equal) is $T = x^{(1-\delta)/(k+1-k\delta)}$, which gives

$$\sum_{n \le x} d_k(n) = xP_k(\log x) + O_{\delta,\varepsilon,k}\bigl(x^{(k-(k-1)\delta)/(k+1-k\delta)+\varepsilon}\bigr);$$

and since the exponent is an increasing function of both δ and ε, choosing them sufficiently small in terms of η and k converts this into

$$\sum_{n \le x} d_k(n) = xP_k(\log x) + O_{\eta,k}\bigl(x^{k/(k+1)+\varepsilon}\bigr).$$