Math 539—Suggested Problems #1

posted Thursday, January 16, 2025

1. Show that the ordered pair (σ_a, σ_c), involving the abscissa of absolute convergence and abscissa of convergence of the same Dirichlet series, can take any value in $\{(u,v) \in \mathbb{R}^2 \colon v \le u \le v+1\}$ (in other words, any value not ruled out by Theorem 1.4). (Hint: consider linear combinations of $\zeta(s)$ and $\eta(s) = \sum_{n=1}^\infty (-1)^{n-1} n^{-s}$, and of their shifts $\zeta(s+c)$ and $\eta(s+c)$.)

2. Let f be a multiplicative function. We would like to have conditions under which we can conclude that both expressions

$$\sum_{n=1}^\infty f(n) \quad \text{and} \quad \prod_p \bigl(1 + f(p) + f(p^2) + \cdots\bigr) \quad \text{converge to equal values.} \qquad (*)$$

We know (Theorem 1.9) that assuming $\sum_{n=1}^\infty |f(n)| < \infty$ is one hypothesis that is sufficient to imply $(*)$.

(a) Prove that assuming

$$\prod_p \bigl(1 + |f(p)| + |f(p^2)| + \cdots\bigr) < \infty$$

is also sufficient to imply $(*)$.

(b) Show that assuming

$$\prod_p \bigl(1 + |f(p) + f(p^2) + \cdots|\bigr) < \infty$$

is *not* sufficient to imply $(*)$.

3. Let $s(x)$ be any function defined on the interval $[0,1]$, and define

$$F(n) = \sum_{1 \le k \le n} s\bigl(\tfrac{k}{n}\bigr) \qquad \text{and} \qquad G(n) = \sum_{\substack{1 \le k \le n \\ (k,n)=1}} s\bigl(\tfrac{k}{n}\bigr).$$

(a) Prove that $G = F * \mu$.

(b) Evaluate the sum of the nth roots of unity

$$\sum_{1 \le k \le n} e^{2\pi i k/n}$$

as a function of n.

(c) Evaluate the sum of the primitive nth roots of unity

$$\sum_{\substack{1 \le k \le n \\ (k,n)=1}} e^{2\pi i k/n}$$

as a function of n.

4. Prove that the following identities all hold in suitable half-planes (be explicit about which half-planes).

(a) $\displaystyle\sum_{\substack{n\geq 1\\(n,q)=1}} \frac{1}{n^s} = \zeta(s)\prod_{p|q}\left(1-\frac{1}{p^s}\right)$ for any positive integer q

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{d(n^2)}{n^s} = \frac{\zeta(s)^3}{\zeta(2s)}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s} = \zeta(s)\zeta(s-a)$ for any complex number a, where $\sigma_a(n) = \sum_{d|n} d^a$ is the generalized sum-of-divisors function.

For the next problem, you may use the fact (which we will prove soon) that $\sum_p \frac{1}{p}$ diverges, where the sum is taken over all primes p.

5. Define the Dirichlet series $P(s) = \sum_p \frac{1}{p^s}$ and $W(s) = \sum_{n=1}^{\infty} \frac{\omega(n)}{n^s}$, where $\omega(n)$ is the number of distinct prime factors of n.

(a) What is the abscissa of convergence for $P(s)$?
(b) Prove that formally, $W(s) = \zeta(s)P(s)$.
(c) What is the abscissa of convergence for $W(s)$?
(d) Prove that $\sum_{n\leq x} \omega(n) \ll_\epsilon x^{1+\epsilon}$ for every $\epsilon > 0$.
(e) Can $W(s)$ be analytically continued to an entire function?

6. Montgomery & Vaughan, Section 1.2, p. 18, #5

7.

(a) Montgomery & Vaughan, Section 1.3, pp. 28–29, #11(c). Recall that Euler's constant C_0 is defined in Corollary 1.15.
(b) Show that $\sum_{n=1}^{\infty}(-1)^n n^{-1+2020\pi i/\log 2} = 0$. (In analytic number theory, $\log$ always denotes the natural logarithm.)