Math 539—Suggested Problems #2
posted Thursday, February 6, 2025

1. Let $a < b$ be real numbers, and let f and g be real-valued functions.

   (a) Suppose that $g(x) \ge 0$ and that $f(x) = o(g(x))$. Give an example to show that it is not necessarily true that $\int_a^x f(u)\,du = o\big(\int_a^x g(u)\,du\big)$. Show however that it is true if we assume that $\int_a^\infty g(u)\,du$ diverges.

   (b) Suppose that the Riemann–Stieltjes integrals $\int_a^b f(u)\,dg(u)$ and $\int_a^b |f(u)|\,dg(u)$ exist. Give an example to show that it is not necessarily true that $\big|\int_a^b f(u)\,dg(u)\big| \le \int_a^b |f(u)|\,dg(u)$. Show however that it is true if we assume that $g(u)$ is increasing.

2. Recall that the *(natural) density* of a set S of positive integers is the following limit (if it exists):

$$\lim_{x\to\infty} \frac{\#\{n \le x \colon n \in S\}}{x} = \lim_{x\to\infty} \bigg(\frac{1}{x} \sum_{\substack{n\le x \\ n\in S}} 1 \bigg).$$

Define the *logarithmic density* of a set S of positive integers to be the following limit (if it exists):

$$\lim_{x\to\infty} \bigg(\frac{1}{\log x} \sum_{\substack{n\le x \\ n\in S}} \frac{1}{n} \bigg).$$

   (a) Suppose that the density of a set S exists and equals c. Show that the logarithmic density of S also exists and equals c.

   (b) Let S_3 be the set of all positive integers whose first (leftmost) digit is 3. Show that the density of S_3 does not exist.

   (c) Show that the logarithmic density of the set S_3 from part (b) does exist, and calculate it.

3.

   (a) Using our Dirichlet convolution method (see page 35 of Montgomery & Vaughan), show that $\displaystyle\sum_{d\le x} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor = 1$ for all $x \ge 1$.

   (b) Using part (a), show that $\displaystyle\bigg| \sum_{d\le x} \frac{\mu(d)}{d} \bigg| \le 1$ for all $x \ge 1$.

   (c) For every real number $t \in (0,1)$, show that $\displaystyle\bigg| \sum_{d\le x} \frac{\mu(d)}{d} t^d \bigg| \le t$ for all $x \ge 1$.

4. Define $\Phi(s) = \sum_{n-1}^\infty \phi(n) n^{-s}$.

   (a) Show directly from the definition of σ_c that the abscissa of convergence of $\Phi(s)$ is $\sigma_c = 2$.

   (b) Can $\Phi(s)$ be analytically continued to a neighborhood of $s = 2$? (You should be able to answer this part without using part (c) below.)

   (c) For $\sigma > 2$, write $\Phi(s)$ in terms of the Riemann zeta function.

5. The goal of this problem is to use Dirichlet's hyperbola method to find a strong asymptotic formula for the summatory function of $2^{\omega(n)}$.

(a) Show that $2^{\omega(n)} = (1 * \mu^2)(n)$.
(b) Let $Q(x)$ be the number of squarefree integers up to x, and define $R(x) = Q(x) - \frac{6}{\pi^2}x$. Find a constant M such that

$$\sum_{n \le x} \frac{\mu^2(n)}{n} = \frac{6}{\pi^2} \log x + M + O\biggl(\frac{1}{\sqrt{x}}\biggr).$$

(Your definition of M might contain an integral, over an infinite interval, of a function involving $R(x)$.)
(c) Find an asymptotic formula for $\sum_{n \le x} 2^{\omega(n)}$ with an error term of the form $O(x^\alpha)$ for the smallest constant $\alpha < 1$ you can obtain. (Your asymptotic formula might involve the constant M from part (b).)

6. Recall that $\sigma(n)$ denotes the sum of the (positive) divisors of n.

(a) Find the smallest constant S such that $\sigma(n) < Sn \log\log n + O(n)$ for all numbers n.
(b) Find the smallest constant T such that $\sigma(n) \le Tn^{21/20}$ for all numbers n.
(c) Are there finitely many or infinitely many numbers n for which $\sigma(n) \ge n^{21/20}$?

7. For this question, you can use the product formula

$$\frac{\sin \pi z}{\pi z} = \prod_{k=1}^{\infty} \biggl(1 - \frac{z^2}{k^2}\biggr) \tag{1}$$

(due to Euler), which is valid for all complex numbers z once the apparent singularity at $z = 0$ is removed; you may also use the Maclaurin series for $\sin z$ and $\cos z$ that you know from calculus. For this question, define $T(z) = \pi z \cot \pi z$.

(a) Logarithmically differentiate the identity (1) to discover an infinite series that sums to $T(z)$.
(b) For $|z| < 1$, conclude that $T(z) = 1 - 2\sum_{n=1}^{\infty} \zeta(2n) z^{2n}$, where ζ is the Riemann zeta-function. (Hint: in your answer to part (a), write $1/(1 - \frac{z^2}{k^2})$ as its Maclaurin series, and exchange the order of summation. Why is that exchange justified?)
(c) Prove that $\zeta(2) = \frac{\pi^2}{6}$. (Hint: write $T(z) = (\cos \pi z) / \frac{\sin \pi z}{\pi z}$ to find a second way to compute the first few terms of its Maclaurin series.)
(d) Show that $zT'(z) = T(z) - T(z)^2 - \pi^2 z^2$.
(e) For all $n > 1$, conclude that

$$\frac{2n+1}{2} \zeta(2n) = \sum_{k=1}^{n-1} \zeta(2k) \zeta(2n - 2k).$$

(f) Calculate $\sum_{n=1}^{\infty} \frac{1}{n^{12}}$ in closed form.

8. By “the $n \times n$ multiplication table” we mean the $n \times n$ array whose (i, j)-th entry is ij. Note that the $n \times n$ multiplication table has n^2 entries, each of which is a positive integer not exceeding n^2, but there are repetitions due to commutativity and to multiple factorizations of various entries.

Define $D(n)$ to be the number of *distinct* integers in the $n \times n$ multiplication table. Erdős gave an ingenious argument showing that $D(n) = o(n^2)$. The idea is as follows: by the Hardy–Ramanujan Theorem, almost all integers up to n have about $\log \log n$ prime factors. That means that almost all of the entries in the $n \times n$ multiplication table have about $2 \log \log n$ prime factors. But these entries do not exceed n^2, and almost all integers up to n^2 only have about $\log \log n^2 = \log \log n + \log 2$ prime factors. Therefore almost all integers up to n^2 must be missing from the $n \times n$ multiplication table.

Turn this sketch into a rigorous, quantitative proof: find an explicit function $f(n)$, satisfying $f(n) = o(n^2)$, for which you can prove that $D(n) \ll f(n)$. Hint: use Theorem 2.12.