Math 539—Suggested Problems #4

posted Thursday, March 13, 2025

1.

(a) Montgomery & Vaughan, Section 10.1, p. 336, #8

(b) Montgomery & Vaughan, Section 10.1, p. 336, #9

(c) Prove that for all $s \in \mathbb{C} \setminus \{1\}$,

$$\zeta(s) = \frac{1}{2(s-1)} \left(\frac{2\pi}{e}\right)^s \prod_{\substack{w\in\mathbb{C} \\ \zeta(w)=0}} \left(1 - \frac{s}{w}\right) e^{s/w}.$$

Note that this product runs over *all* the zeros of $\zeta(s)$, as opposed to our usual convention of products $\prod_\rho$ that run only over the nontrivial zeros of $\zeta(s)$. Hint: combine Theorem 10.12 and equation (C.1).

2. Montgomery & Vaughan, Section 10.2, p. 353, #2

3. Let $k_1, \dots, k_j$ be positive even integers. Prove that

$$\sum_{n_1=1}^{\infty} \cdots \sum_{\substack{n_j=1 \\ (n_1,\dots,n_j)=1}}^{\infty} \frac{1}{n_1^{k_1} \cdots n_j^{k_j}}$$

is a rational number. (You may use Corollary B.3)

4. In both parts of this problem, Θ denotes the supremum of the real parts of all zeros of $\zeta(s)$, so that $\frac{1}{2} \le \Theta \le 1$. (This problem is valid, but worse than trivial, if it happens that $\Theta = 1$.)

(a) Montgomery & Vaughan, Section 13.1, p. 430, #1. Note that this problem is a generalization of Theorem 13.1.

(b) Show that there exists a constant $C > 0$ such that for every prime p, the next prime after p is at most $p + Cp^\Theta \log^2 p$.

5. The upper bound in equation (13.1) of Montgomery & Vaughan can be strengthened to an asymptotic formula if we use the results of Chapter 14. For this problem, $\rho = \beta + i\gamma$ denotes a nontrivial zero of $\zeta(s)$.

(a) Prove that $\dfrac{1}{|\rho|} = \dfrac{1}{|\gamma|} + O\left(\dfrac{1}{|\gamma|^3}\right)$ uniformly for all ρ.

(b) Prove that

$$\sum_{\substack{\rho \\ |\gamma|\le T}} \frac{1}{|\rho|} = \frac{1}{2\pi} \log^2 T - \frac{\log 2\pi}{\pi} \log T + O(1).$$

Hint: write the left-hand side as a Riemann–Stieltjes integral of the form $\int \dots \, dN(t)$, and use Corollary 14.3.

6. Assume the Riemann hypothesis. Let $x \geq 4$ and $\varepsilon > 0$. Show that

$$\psi(x) = x - \sqrt{x} \sum_{\substack{\rho \in \mathbb{C} \\ 0 < \gamma < x^{1/2+\varepsilon}}} 2\Re\left(\frac{x^{\rho - 1/2}}{\rho}\right) + O_\varepsilon(x^{1/2-\varepsilon/2}),$$

where the sum is over nontrivial zeros of $\zeta(s)$ with the usual convention $\rho = \beta + i\gamma$. Conclude that for $y \geq 2$,

$$\frac{\psi(e^y) - e^y}{e^{y/2}} = - \sum_{\substack{\rho \in \mathbb{C} \\ 0 < \gamma < x^{1/2+\varepsilon}}} \frac{8\gamma \sin(\gamma y) + 4\cos(\gamma y)}{1 + 4\gamma^2} + o_\varepsilon(1).$$

(This is a Fourier-like expansion of a normalized version of the error term for $\psi(x)$, which is not a periodic function but is "almost-periodic"—it repeats its values up to arbitrarily small errors, with the "period" depending on the prescribed maximum error.)