Math 539—Suggested Problems #5
posted Tuesday, April 1, 2025

1. Write down (with justification) all Dirichlet characters to the modulus 7; to the modulus 15; to the modulus 16.

2. Let p be a prime, and let δ denote the characteristic function of the set of integers that are primitive roots (mod p). Note that δ is periodic with period p and supported on integers relatively prime to p, and so we immediately know that there exist constants c_χ such that

$$\delta(n) = \sum_{\chi \pmod p} c_\chi \chi(n)$$

for every integer n. Prove that $c_\chi = \frac{1}{p-1} \sum_g \chi(g)$, where the sum is taken over all primitive roots $g \pmod p$.

3. Montgomery & Vaughan, Section 4.2, p. 119, #2–5. Hint: a sum of the form $\sum_y \big| \sum_z \cdots \big|^2$ can be written as a triple sum $\sum_y \big(\sum_{z_1} \cdots \big) \overline{\big(\sum_{z_2} \cdots \big)}$.

4.

(a) Montgomery & Vaughan, Section 4.3, pp. 127, #1
(b) Montgomery & Vaughan, Section 4.3, pp. 128, #5(a),(b)

For the next problem, it will help you to use the *Brun–Titchmarsh theorem* (which is Theorem 3.9 in Montgomery & Vaughan), especially the case $x = 0$ in the given notation. The Brun–Titchmarsh theorem serves as a sort of replacement for Chebyshev's upper bound for $\pi(y)$.

5. Let $x \ge 3$ be a real number, let q be a positive integer, and let a be an integer, with $0 < a < q$, that is relatively prime to q.

(a) Show that

$$\sum_{\substack{p \le x \\ p \equiv a \pmod q}} \frac{1}{p} \ll \frac{1}{a} + \frac{\log\log x}{\phi(q)}, \tag{1}$$

where the implicit constant is absolute (doesn't depend on x, a, or q.) Show that the term $\frac{1}{a}$ on the right-hand side may be omitted if a is not prime. (Hint: Brun–Titchmarsh and partial summation.)

(b) Show that the term $\frac{1}{a}$ on the right-hand side of the estimate (1) may not be omitted if a is prime.

6. For every finite abelian group G, show that there exist infinitely many integers n for which G is isomorphic to a subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$.