AN ANNOTATED BIBLIOGRAPHY FOR COMPARATIVE PRIME NUMBER THEORY

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Abstract. ...

1. Introduction

[basically a longer version of the abstract, together with notes on the organization of this article]

2. Notation

[gathering together notation and other information that will be useful for the entire annotated bibliography]

The letter p will always denote a prime.

2.1. **Elementary functions.**

Euler ϕ-function. ω and Ω and μ and Λ.

$\lambda(n) = (-1)^{\Omega(n)}$ is Liouville's function.

Also $c_q(a) = \#\{b \pmod q \colon b^2 \equiv a \pmod q\}$. Shorthand: $c_q = c_q(1)$, which is also the number of real characters $\pmod q$, or equivalently the index $\big[(\mathbb{Z}/q\mathbb{Z})^\times : \big((\mathbb{Z}/q\mathbb{Z})^\times\big)^2\big]$. It is easy to see, for $(a,q)=1$, that $c_q(a)$ equals c_q if a is a square $\pmod q$ and 0 otherwise.

Logarithmic integrals:

$$\mathrm{li}(x) = \lim_{\varepsilon\to0+}\left(\int_0^{1-\varepsilon}\frac{dt}{\log t} + \int_{1+\varepsilon}^x \frac{dt}{\log t}\right)$$
$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} = \mathrm{li}(x) - \mathrm{li}(2)$$

These functions have asymptotic expansions, one example of which is

$$\mathrm{li}(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6x}{\log^4 x} + O\bigg(\frac{x}{\log^5 x}\bigg).$$

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2.2. **Prime counting functions.** Prime counting functions:

$$\pi(x) = \#\{p \le x\} = \sum_{p \le x} 1$$

$$\Pi(x) = \sum_{n \le x} \frac{\Lambda(n)}{\log n} = \sum_{p^k \le x} \frac{1}{k} = \sum_{k=1}^{\infty} \frac{1}{k}\pi(x^{1/k})$$

$$\Pi^*(x) = \sum_{p^k \le x} 1 = \sum_{k=1}^{\infty} \pi(x^{1/k})$$

$$\theta(x) = \sum_{p \le x} \log p$$

$$\psi(x) = \sum_{n \le x} \Lambda(n) = \sum_{p^k \le x} \log p = \sum_{p \le x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p = \sum_{k=1}^{\infty} \frac{1}{k}\theta(x^{1/k})$$

2.3. **Primes in arithmetic progressions.** Counting functions for primes in arithmetic progressions:

$$\pi(x;q,a) = \#\{p \le x \colon p \equiv a \pmod q\} = \sum_{\substack{p \le x \\ p \equiv a \pmod q}} 1$$

$$\theta(x;q,a) = \sum_{\substack{p \le x \\ p \equiv a \pmod q}} \log p$$

$$\psi(x;q,a) = \sum_{\substack{n \le x \\ n \equiv a \pmod q}} \Lambda(n) = \sum_{\substack{p^k \le x \\ p^k \equiv a \pmod q}} \log p$$

General notation for racing (multi)sets of residue classes; show that the case of different moduli reduces to this.... Some relatives, when q is a modulus with primitive roots:

$$\pi(x;q,R) = \#\{p \le x \colon p \text{ is a quadratic residue} \pmod q\}$$

$$\pi(x;q,N) = \#\{p \le x \colon p \text{ is a quadratic nonresidue} \pmod q\}$$

In these arithmetic progression functions, an extra argument denotes a difference (some authors use Δ for this): for example,

$$\psi(x;q,a,1) = \psi(x;q,a) - \psi(x;q,1) \quad \text{and} \quad \pi(x;q,N,R) = \pi(x;q,N) - \pi(x;q,R).$$

2.4. **Error terms for prime counting functions.** We use Δ for the error terms in prime counting functions:

$$\Delta^{\psi}(x) = \psi(x) - x, \quad \Delta^{\theta}(x) = \theta(x) - x, \quad \Delta^{\Pi}(x) = \Pi(x) - \mathrm{li}(x), \quad \Delta^{\pi}(x) = \pi(x) - \mathrm{li}(x).$$

(We're not very careful about the difference between li and Li here.) We also use E for normalized versions of these error terms:

$$E^{\psi}(x) = \frac{\Delta^{\psi}(x)}{\sqrt{x}}, \quad E^{\theta}(x) = \frac{\Delta^{\theta}(x)}{\sqrt{x}}, \quad E^{\Pi}(x) = \frac{\Delta^{\Pi}(x)}{\sqrt{x}/\log x}, \quad E^{\pi}(x) = \frac{\Delta^{\pi}(x)}{\sqrt{x}/\log x}.$$

It's not uncommon to integrate these error terms: for any $f \in \{\pi, \Pi, \theta, \psi\}$ we define $\Delta_0^f(x) = \Delta^f(x)$ and, for $n \ge 1$,

$$\Delta_n^f(x) = \int_2^x \Delta_{n-1}^f(x)\,dx.$$

We also define $\Delta^f_{|0|}(x) = |\Delta^f(x)|$ and $\Delta^f_{|n|}(x) = \frac{1}{x}\int_2^x \Delta^f_{|n-1|}(x)\,dx$. There are similar logarithmic integration operators: we define $\not\Delta^f_0(x) = \Delta^f(x)$ and, for $n \ge 1$,

$$\not\Delta^f_n(x) = \int_2^x \not\Delta^f_{n-1}(x)\,\frac{dx}{x}.$$

[for summatory functions, the Δ operator multiplies each factor by $x-n$, while the $\not\Delta$ operator multiplies each factor by $\log\frac{x}{n}$; for explicit formulae, the Δ operator changes x^ρ/ρ to $x^{\rho+1}/\rho(\rho+1)$ and so on, while the $\not\Delta$ operator multiplies each x^ρ term by $1/\rho$]

When we count primes in arithmetic progressions, we multiply through by $\phi(q)$ for simplicity—for example,

$$\Delta^\psi(x;q,a) = \phi(q)\psi(x;q,a) - x \quad\text{and}\quad \Delta^\pi(x;q,a) = \phi(q)\pi(x;q,a) - \mathrm{li}(x).$$

Then the normalization is the same as before—for example,

$$E^\psi(x;q,a) = \frac{\Delta^\psi(x;q,a)}{\sqrt{x}} \quad\text{and}\quad E^\pi(x;q,a) = \frac{\Delta^\pi(x;q,a)}{\sqrt{x}/\log x}.$$

VARIANT $\mathring{\Delta}$ WHERE WE SUBTRACT for example $\pi(x)$ instead of $\mathrm{li}(x)$? We extend our convention regarding counting functions in arithmetic progressions, so that for example,

$$\Delta^\psi(x;q,a,b) = \Delta^\psi(x;q,a) - \Delta^\psi(x;q,b) \quad\text{and}\quad E^\pi(x;q,a,b) = E^\pi(x;q,a) - E^\pi(x;q,b).$$

Notice that the first such function is almost redundant, since $\Delta^\psi(x;q,a,b) = \phi(q)\psi(x;q,a,b)$ exactly. (And recall that some authors use Δ to mean this difference function without the factor $\phi(q)$, which we are calling ψ here.) However, there will situations where each notation is useful to us; furthermore, this new use of Δ already follows from existing notational conventions.

We'll also have some notation for vector-valued versions of these functions; we'll need to do it carefully to make the $r=2$ case of the notation not clash with the difference notations above. [See [89] for an example.]

2.5. **Weighted versions and variants.** Interval notation like $\pi\big([x,2x)\big)$....

We use various subscripts to indicate weighted versions of the above sums.

- The subscript 0 modifies jump discontinuities....
- The subscript r represents one of the above sums weighted by a reciprocal factor (often resulting in a "Mertens sum"); for example,

$$\pi_r(x) = \sum_{p\le x}\frac{1}{p} \quad\text{and}\quad \psi_r(x;q,a) = \sum_{\substack{n\le x\\ n\equiv a \pmod q}}\frac{\Lambda(n)}{n}.$$

  If we need both the 0 and r subscripts, we'll simply write (for example) $\pi_{0r}(x)$.
- The subscript e represents one of the above sums weighted by an exponentially decaying factor rather than cutting off abruptly at x; for example,

$$\pi_e(x) = \sum_p e^{-p/x} \quad\text{and}\quad \psi_e(x;q,a) = \sum_{\substack{n\ge 1\\ n\equiv a \pmod q}}\Lambda(n)e^{-n/x}.$$

  In terms of their asymptotics, these exponentially weighted sums usually act like their abrupt-cutoff versions; for example, $\pi_e(x)$ sometimes acts like $\pi(x)$. However, their oscillations are typically damped, often resulting in rather different properties when comparing two such functions to each other (such as the exponentially weighted version having a bias for one sign while the unweighted version exhibits oscillations of sign).

- The (somewhat arbitrary) subscript l represents one of the above sums weighted by a strange-looking factor: by way of example,

$$\pi_l(x,r) = \sum_p e^{-\frac{1}{r}(\log\frac{p}{x})^2} \quad\text{and}\quad \psi_l(x,r;q,a) = \sum_{\substack{n\ge1 \\ n\equiv a \pmod q}} \Lambda(n) e^{-\frac{1}{r}(\log\frac{n}{x})^2}.$$

  This can be thought of as restricting the range of summation to approximately $[e^{-\sqrt{r}}x, e^{\sqrt{r}}x]$. Remarks from the above version apply here as well to the asymptotics and comparative properties of this version.
- When a weight function is a Dirichlet charater χ (see Section 3.1), we follow the tradition of putting χ as an extra argument rather than a subscript; for example,

$$\theta(x,\chi) = \sum_{p\le x} \chi(p)\log p.$$

2.6. **Summatory functions.** Summatory functions: Mertens sum $M(x) = \sum_{n\le x}\mu(n)$ and the related $L(x) = \sum_{n\le x}\lambda(n)$. Arithmetic progression notation in play here too, for example

$$M(x;q,a) = \sum_{\substack{n\le x \\ n\equiv a \pmod q}} \mu(n).$$

ADD M_e example. Mertens and Polyá conjectures.

2.7. **Counting sign changes.** If f, g, and h are functions from $(1,\infty)$ to $\mathbb{R}$, then we define $W(h;T)$ to be the number of sign changes of $h(x)$ in the interval $(1,T)$, while $W(f,g;T) = W(f-g;T)$ similarly counts sign changes of the difference $f(x)-g(x)$. Certain special cases of this notation get a shorthand: we define $W^\pi(T) = W(\pi,\mathrm{li};T)$ and $W^\Pi(T) = W(\Pi,\mathrm{li};T)$ and $W^\theta(T) = W(\theta,x;T)$ (where x denotes the identity function) and $W^\psi(T) = W(\psi,x;T)$. (We aren't very careful about the difference between $\mathrm{li}(x)$ and $\mathrm{Li}(x)$ here.) And we further shorten $W(T) = W^\pi(T)$.

In addition, let q be a positive integer, and let a and b be distinct reduced residues (mod q). Then we define $W^f_{q;a,b}(T) = W(f(x;q,a), f(x;q,b);T)$, for any function f for which $f(x;q,a)$ makes sense (such as $f = \pi, \Pi, \Pi^*, \theta, \psi$). As shorter shorthand, $W_{q;a,b}(T) = W^\pi_{q;a,b}(T) = W(\pi(x;q,a), \pi(x;q,b);T)$.

To be pedantic,

$$W(h;T) = \max\big\{n\ge 0\colon \text{there exist } 1<t_0<t_1<\cdots<t_n<T \\ \text{with } h(t_{j-1})h(t_j)<0 \text{ for all } 1\le j\le n\big\}.$$

We can demand large oscillations to go along with our sign changes by adding a function as an additional argument:

$$W\big(h;T;S\big) = \max\big\{n\ge 0\colon \text{there exist } 1<t_0<t_1<\cdots<t_n<T \\ \text{with } h(t_{j-1})h(t_j)<0 \text{ for all } 1\le j\le n \text{ and } |h(t_j)|>S \text{ for all } 0\le j\le n\big\}.$$

This additional argument can be used with the notations above, such as $W_{q;a,b}(T;S)$.

2.8. **Densities.** Densities: the natural density of a set S of positive real numbers is

$$\lim_{x\to\infty}\frac{\text{measure}\big(\{0<t\le x\colon x\in S\}\big)}{x} = \lim_{x\to\infty}\frac{1}{x}\int_{\substack{0<t<x \\ x\in S}} dt.$$

On the other hand, the logarithmic density of a set $S \subset (1,\infty)$ is

$$\delta(S) = \lim_{x\to\infty} \frac{1}{\log x} \int_{\substack{1<t<x \\ x\in S}} \frac{dt}{t}.$$

An easy change of variables shows that the logarithmic density of S equals the natural density of the set $\log S = \{\log z \colon z \in S\}$. Moreover, a partial summation argument shows that if the natural density of S exists, then the logarithmic density $\delta(S)$ also exists and has the same value. However, there are sets whose natural density does not exist but whose logarithmic density does exist; for example, the union (over $k \in \mathbb{N}$) of all the intervals of the form $[10^{2k-1}, 10^{2k})$ has this property. (There are discrete analogues of natural and logarithmic density for subsets of the positive integers; this last counterexample is analogous to the set of all positive integers with an even number of digits, which does not have a natural density but whose logarithmic density equals $\frac{1}{2}$.)

We will use many variants of this logarithmic density notation. If $f_1, \dots, f_r$ are functions from $(1,\infty)$ to $\mathbb{R}$, then we define the shorthand notation

$$\delta(f_1, f_2, \dots, f_r) = \delta\big(\{x > 1 \colon f_1(x) > f_2(x) > \cdots > f_r(x)\}\big).$$

For example, $\delta(\mathrm{li}, \pi)$ is the logarithmic density of the set of real numbers $x > 1$ for which $\mathrm{li}(x) > \pi(x)$. Certain special cases of this notation get even further shorthand. For example, let q be a positive integer, and let $a_1, \dots, a_r$ be distinct reduced residues (mod q). Then we define

$$\delta_{q;a_1,\dots,a_r} = \delta\big(\pi(x;q,a_1), \dots, \pi(x;q,a_r)\big) = \delta\big(\{x > 1 \colon \pi(x;q,a_1) > \cdots > \pi(x;q,a_r)\}\big).$$

We also define

$$\delta_{q;N,R} = \delta\big(\pi(x;q,N), \pi(x;q,R)\big) = \delta\big(\{x > 1 \colon \pi(x;q,N) > \pi(x;q,R)\}\big)$$

and similarly for $\delta_{q;R,N}$.

Finally, we define the upper and lower logarithmic densities of S (which always exist) to be

$$\overline{\delta}(S) = \limsup_{x\to\infty} \frac{1}{\log x} \int_{\substack{1<t<x \\ x\in S}} \frac{dt}{t}, \quad \underline{\delta}(S) = \liminf_{x\to\infty} \frac{1}{\log x} \int_{\substack{1<t<x \\ x\in S}} \frac{dt}{t},$$

so that $\delta(S)$ exists if and only if $\overline{\delta}(S) = \underline{\delta}(S)$. This notation for upper and lower densities can propogate through our shorthand notations as well; for instance, $\underline{\delta}_{q;N,R} = \underline{\delta}\big(\{x > 1 \colon \pi(x;q,N) > \pi(x;q,R)\}\big)$.

2.9. **Limiting distribution and density functions.** [Limiting cumulative distribution functions and limiting density functions (define), and their logarithmic counterparts.] Given a function $h \colon [0,\infty) \to \mathbb{R}$, the limiting (or asymptotic) cumulative distribution function of h is the nondecreasing function

$$\kappa(\alpha) = \lim_{T\to\infty} \frac{\mathrm{meas}\{t \in [0,T] \colon h(t) < \alpha\}}{T},$$

which we require to be defined except for at most a countable number of jump discontinuities. Given such a distribution function κ, its spectrum is the set of points $\alpha \in \mathbb{R}$ such that $\kappa(x) < \kappa(y)$ for all $x < \alpha < y$.

3. Complex analytic stuff

3.1. **Dirichlet characters and Dirichlet L-functions.** Dirichlet characters, χ_0, χ_D, χ^* (χ_1 and β_1?)....

$N(T)$ and $N(T,\chi)$, and the convention $\rho = \beta + i\gamma$

3.2. **Landau's theorem.** For a real-valued function $A(x)$, define

$$g(s) = \int_1^\infty \frac{A(x)}{x^s}\,dx$$

Suppose that $g(\sigma)$ is analytic on the ray $\{\sigma \in \mathbb{R} \colon \sigma > \gamma\}$ but $g(s)$ is not analytic in the half-plane $\{s \in \mathbb{C} \colon \sigma > \gamma\}$. Then $A(x)$ has arbitrarily large sign changes.

3.3. **Explicit formulas.** [my favorite formulas!]

—for now, just two such formulas: the fundamental explicit formula

$$\psi_0(x) = x - \sum_{\substack{\rho\in\mathbb{C} \\ 0<\Re\rho<1 \\ \zeta(\rho)=0}} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log\left(1-\frac{1}{x^2}\right),$$

and a homework problem from the recent MATH 539: on GRH, for $(a,q)=(b,q)=1$,

$$\frac{\theta(x;q,a)-\theta(x;q,b)}{\sqrt{x}} = \frac{c_q(b)-c_q(a)}{\phi(q)} - \frac{1}{\phi(q)} \sum_{\chi \pmod q} \big(\overline{\chi}(a)-\overline{\chi}(b)\big) \lim_{T\to\infty} \sum_{\substack{|\gamma|\le T \\ L(1/2+i\gamma,\chi)=0}} \frac{x^{i\gamma}}{\frac12+i\gamma} + O_q(x^{-1/6}),$$

where $c_q(r)$ is the number of solutions to $x^2 \equiv r \pmod q$.

[In terms of explicit formulas/limiting distributions: the normalized error terms of ψ and Π are centered at 0, while those of θ and π are centered at -1 (and that of Π^* is centered at 1). The normalized error term of $\theta(x;q,a,b)$, above, is centered at a positive value of a is a quadratic nonresidue and b is a quadratic residue, centered at a negative value if vice versa, and centered at 0 if a and b are both quadratic residues or both quadratic nonresidues.]

3.4. **The power sum method.** Brief description of both "one-sided" and "two-sided" Turán theorems

3.5. **k-functions.** For $\Im z > 0$, define

$$k(z,\chi) = \sum_{\gamma>0} e^{\rho z} \quad\text{and}\quad K(z,\chi) = \sum_{\gamma>0} \frac{e^{\rho z}}{\rho},$$

where the sums are over zeros of $L(s,\chi)$. Let $\mathcal{M}$ be the Riemann surface for $\log z$; every point on the surface can be uniquely written as re^{ia} where $a \in \mathbb{R}$. Let z^c denote the natural extension of complex conjugation to $\mathcal{M}$, namely $(re^{ia})^c = re^{-ia}$; also let z^* denote an extension of multiplication by -1 to $\mathcal{M}$, namely $(re^{ia})^* = re^{i(a-\pi)}$.

Define

$$D(z,\chi) = -\sum_{\substack{\beta>0 \\ L(\beta,\chi)=0}} e^{\beta z} + \frac{1}{e^{2z}-1}\begin{cases} e^{3z}+e^{2z}-1, & \text{if } \chi=\chi_0, \\ e^{z}, & \text{if } \chi\ne\chi_0 \text{ and } \chi(-1)=1, \\ e^{2z}, & \text{if } \chi(-1)=-1. \end{cases}$$

Also define

$$F(x,\chi) = \lim_{y\to0^+}\Big(K(x+iy,\chi) + \overline{K(x+iy,\overline{\chi})}\Big).$$

Also also define

$$R_1(x) = \frac{1}{2}\log(1-e^{-2x}), \quad R_{-1}(x) = \frac{1}{2}\log\frac{e^x-1}{e^x+1}.$$

Define constants ($B(\chi)$ probably conflicts with other notation)

$$B(\chi) = \sum_{\substack{\beta>0 \\ L(\beta,\chi)=0}} \frac{1}{\beta} - \frac{C_0}{2} - \frac{1}{2}\log\frac{\pi}{q} + F(0,\chi) - \begin{cases} 1, & \text{if } \chi=\chi_0, \\ 0, & \text{if } \chi\neq\chi_0 \text{ and } \chi(-1)=1, \\ \log 2, & \text{if } \chi(-1)=-1. \end{cases}$$

$$C(\chi) = B(\chi) + C_0 + \log\frac{2\pi}{q}.$$

3.6. **Hypotheses on zeros.** [various assumptions on the zeros of Dirichlet L-functions, including GRH and LI. LI(σ) is LI for the set of imaginary parts corresponding to zeros with real parts at least σ]

- HC: the Haselgrove condition that $L(\sigma,\chi)$ does not vanish on the segment $0<\sigma<1$ of the real axis. By continuity, this means there exists $E_k>0$ such that no $L(s,\chi)$ with $\chi \pmod k$ vanishes on the rectangle $\{0<\sigma<1,\ |t|\le E_k\}$; we use HC($E_k$) if we need to name this parameter. Remark: a zero on the real axis would cause a non-oscillating term (a bias).
- GRH(H) is GRH up to height H (sometimes called the "finite Riemann–Piltz" conjecture, Riemann–Piltz itself being GRH for Dirichlet L-functions), namely the statement that if ρ is a nontrivial zero of $L(s,\chi)$ with $|\gamma|\le H$, then $\sigma=\frac{1}{2}$.

  Note that if $E_k \ge H$, then GRH(H) is actually implied by HC(E_k). On the other hand, GRH(H) gives no constraint at all upon zeros on the critical line. We therefore introduce the notation GRH(H, E_k) to mean the combination of GRH(H) and HC(E_k), the latter of which constrains only the zeros on the critical line when $E_k<H$.

  Note also that GRH(0) is almost the same as HC, except that the former allows for the possibility of a zero at $s=\frac{1}{2}$.
- σ_0-GRH for zero-free strip $\sigma>\sigma_0$.
- GRH is $\frac{1}{2}$-GRH and, simultaneously, GRH(∞).
- Given a nonempty set X of Dirichlet characters (or L-functions), $\Theta(X)$ denotes the supremum of the real parts of their zeros, that is, the smallest real number such that $\Theta(X)$-GRH holds. SA for Supremum Attained ("Ingham's condition"). Special cases: $\Theta(q)$ for all Dirichlet L-functions (mod q), $\Theta(\chi)$ for the single character χ, and Θ alone for $\zeta(s)$.

4. Types of questions

[what do we want to know about these prime counting functions?]

Given two functions $f,g\colon (1,\infty)\to\mathbb{R}$ (for example, $\pi(x)$ and $\mathrm{li}(x)$, or $\pi(x;4,1)$ and $\pi(x;4,3)$) that are asymptotic to each other, we can ask:

- Are there arbitrarily large values of x for which $f(x)>g(x)$, and arbitrarily large values of x for which $g(x)<f(x)$? In other words, does the difference $f(x)-g(x)$ change signs infinitely often? (These are not quite mathematically identical because of the possibility of plentiful or carefully arranged ties $f(x)=g(x)$, so implicit in this question is asking whether such ties are rare.) The other alternative is that one of the functions exceeds the other for all sufficiently large x.
- How large and positive can the difference $f(x)-g(x)$ get? How large and negative can it get?

- More generally, what is the distribution of values of $f(x) - g(x)$? Is it possible that some suitably normalized version of this difference, such as $(f(x)-g(x))/\sqrt{x}$, actually has a limiting distribution or a limiting logarithmic distribution?
- How often does the difference $f(x) - g(x)$ change sign? How many sign changes are there in $(1, X)$ as a function of X? How close can we take $Y = Y(X)$ to X to ensure that there is always a sign change in $[X, Y]$?
- What is the natural density of the set of real numbers $x > 1$ for which $f(x) > g(x)$? What is its logarithmic density, which we denote by $\delta(f, g)$? (As we shall see, we believe that the natural densities of such sets do not exist in prime number races, but that their logarithmic densities do exist.)
- Given a family of races, such as $\pi(x; q, N)$ versus $\pi(x; q, R)$: how do answers to the above questions, such as $\delta_{q;N,R}$, depend upon the member of the family (q in this case)? Do the distributions of the members of the family tend to some limit, such as a normal distribution?

Given several functions $f_1, \dots, f_r \colon (1, \infty) \to \mathbb{R}$, we can ask some of the above questions as well:

- Are there arbitrarily large values of x for which $f_1(x) > \cdots > f_r(x)$?
- More generally, what is the distribution of values of the vector $\big(f_1(x), \dots, f_r(x)\big) \in \mathbb{R}^r$? Is it possible that some suitably normalized version of this difference actually has a limiting distribution or a limiting logarithmic distribution?
- What is the natural density of the set of real numbers $x > 1$ for which $f_1(x) > \cdots > f_r(x)$? What is its logarithmic density, which we denote by $\delta(f, g)$? (As before, we believe that the natural densities of such sets do not exist in prime number races, but that their logarithmic densities do exist.)
- Given a family of such r-way races, how do answers to the above questions depend upon the member of the family? Do the distributions of the members of the family tend to some limit, such as a multivariate normal distribution?

The papers [19] and [28] present organized schema for problems in comparative prime number theory, although several of the questions listed above had not yet been investigated sufficiently deeply to make their lists.

Global LaTeX changes

- change E_k to A_k in the HC references
- change every `\pmod` to `\mod`
- change every `\epsilon` and `\varepsilon` to `\ep`
- change every `\varphi` to `\phi`
- change "cites `\cite`" to "cites∼`\cite`"
- change every Turan and Tur\'{a}n to Tur\'an
- change "zeroes" to "zeros"
- change "paper" to "article"
- unhyphenate "non-"s and "square-free"s
- remove tabs everywhere

5. References

REMARKS ON THE ANNOTATIONS Systemized notation, as described above, rather than preserving the notation of the original articles.

[1] P. Chebyshev, *Lettre de M. le professeur Tchébychev a M. Fuss, sur un nouveau théorème rélatif aux nombres premiers contenus dans la formes* $4n+1$ *et* $4n+3$., Bull. de la Classe phys. math. de l'Acad. Imp. des Sciences St. Petersburg **11** (1853), 208 (French).

In this letter, the author remarks for the first time that there appear to be more primes of the form $4n+3$ than $4n+1$, in that their counting functions "differ notably in their second terms" (original French: *"different notablement entre elles par leurs seconds termes"*). Several assertions are made (without proof): that $\pi(x;4,3,1)/(\sqrt{x}/\log x)$ takes values arbitrarily close to 1; that $\pi_e(x,\chi_{-4}) \to -\infty$ as $x \to \infty$; and that if $f(x)$ is a constantly decreasing function with $\lim_{x\to\infty} x^{1/2} f(x) \neq 0$, then the series $\sum_p \chi_{-4}(p) f(p)$ is not convergent.

[2] P. Phragmén, *Sur le logarithme intégral et la fonction* $f(x)$ *de Riemann*, Öfversigt af Kongl. Vetenskaps–Akademiens Föhandlingar. **48** (1891), 599–616 (French).

The author establishes a general proposition (similar to the eventual "Landau's theorem") capable of establishing that particular functions change sign for arbitrarily large arguments. From this proposition, the author shows that $\Pi(x) - (\mathrm{li}(x) - \log 2)$ changes sign infinitely often, as do the differences $\psi(x) - (x - \log \frac{\pi}{2})$ and $\Pi_r^*(x) - (\log\log x + C_0)$ and $\psi_r(x) - (\log x - C_0)$, where C_0 is Euler's constant. Also, each of the differences $\Pi_r(x;4,1) - \big(\frac12 \,\mathrm{Li}(x) - \frac12 \log \frac{\log x}{\log 2} - \log 2\big)$ and $\Pi_r(x;4,3) - \big(\frac12 \,\mathrm{Li}(x) - \frac12 \log \frac{\log x}{\log 2}\big)$ and $\Pi_r(x;4,1,3) + \log 2$ changes signs infinitely often. Finally, the author establishes the assertion of Chebyshev that 1 is a limit point of the function $\pi(x;4,3,1)/(\sqrt{x}/\log x)$.

This article cites [1].

[3] E. Landau, *Über einen Satz von Tschebyschef*, Mathematische Annalen **61** (1905), 527–550 (German). MR1511360

[4] J. E. Littlewood, *Sur la distribution des nombres premiers*, Comptes Rendus de l'Acad. Sci. Paris **158** (1914), 1869–1872 (French).

[5] G. H. Hardy and J. E. Littlewood, *Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes*, Acta Math. **41** (1916), no. 1, 119–196. MR1555148

This article contains full proofs of several results that had been announced by (at least one of) the authors in the few years prior.

In Section 2.2, the authors obtain an explicit formula for the exponentially weighted sum $\sum_{n=1}^\infty (\Lambda(n) - 1)e^{-n/x} = \psi_e(x) - 1/(e^{1/x} - 1)$; furthermore, assuming RH, they show that this expression is both $\ll \sqrt{x}$ and $\Omega_\pm(\sqrt{x})$. From the latter they deduce that $\psi(x) - x = \Omega_\pm(\sqrt{x})$.

In Section 2.3 they consider the function $\sum_{p\ge3} (-1)^{(p+1)/2} e^{-p/x} = -\pi_e(x,\chi_{-4})$. Assuming GRH for $L(s,\chi_{-4})$, they prove that $\pi_e(x,\chi_{-4}) \to -\infty$ as $x \to \infty$, which is one way of justifying Chebyshev's observation that there are more primes congruent to 3 (mod 4) than to 1 (mod 4).

In Section 5, the authors provide a full proof of "Littlewood's theorem" (announced in [4]) on irregularities in the distribution of primes: they prove that

$$\Delta^\pi(x) = \Omega_\pm\biggl(\frac{\sqrt{x}}{\log x} \log\log\log x\biggr),$$

which in particular refutes the conjecture that $\pi(x) < \mathrm{li}(x)$ for all $x > 1$. Their proof, which begins with the assumption of RH thanks to prior work of Landau [140, Sections 201–3], uses homogenous Diophantine approximation for the imaginary parts of the zeros of $\zeta(s)$. They assert that it can be shown in a similar way that $\psi(x,\chi_{-4}) = \Omega_\pm(\sqrt{x}\log\log\log x)$ and

$$\pi(x;4,3,1) = \Omega_\pm\biggl(\frac{\sqrt{x}}{\log x} \log\log\log x\biggr);$$

these results are actually dissonant with Chebyshev's observations.

(As a side remark, this is also the article in which appears the proof of the asymptotic formula for the second moment of $\zeta(s)$ on the critical line.)

This article cites [4, 137, 140, 142].

[6] E. Landau, *Über einige ältere Vermutungen und Behauptungen in der Primzahltheorie*, Math. Z. **1** (1918), 1–24 (German).

Reportedly, the author shows that Chebyshev's assertion that the function $\pi_e(x, \chi_{-4}) \to -\infty$ as $x \to \infty$ implies GRH for $L(s, \chi_{-4})$.

[7] ———, *Über einige ältere Vermutungen und Behauptungen in der Primzahltheorie. Zweite Abhandlung*, Math. Z. **1** (1918), 213–219 (German). MR1544293

Reportedly, the author provides a simpler proof of the result of Hardy–Littlewood [5, Section 2.3] justifying, assuming GRH for $L(s, \chi_{-4})$, Chebyshev's assertion that the function $\pi_e(x, \chi_{-4}) \to -\infty$ as $x \to \infty$.

[8] G. Pólya, *Über das Vorzeichen des Restgliedes im Primzahltheorie*, Gött. Nachr. (1930), 19–27 (German).

Reportedly, the author proves that $\limsup_{T\to\infty} W^{\psi}(T)/\log T \ge \gamma_1/\pi$, where $\gamma_1 \approx 14.135$ is the smallest ordinate of a nontrivial zero of $\zeta(s)$. It was known for a long while that this article contained an error that could be mended; the corrected version finally appeared as [184].

[9] S. Skewes, *On the difference* $\pi(x) - \mathrm{li}(x)$ *(I)*, J. London Math. Soc. **8** (1933), no. 4, 277–283. MR1573970

This article shows, assuming RH, that $\pi(x) > \mathrm{li}(x)$ for some $x < 10^{10^{10^{34}}}$. Littlewood [4, 5] proved the existence of such an x by considering the function $F(z) = \sum_{\gamma>0} e^{-\gamma(\xi+i\eta)}/\gamma$ for $0 \le \xi \le 1$ and $\eta \ge 1$, which is relevant since the explicit formula yields $-2\Im F(i \log x) = E^{\psi}(x) + O(1)$. Using the Dirichlet box principle, Littlewood showed that $\Im F(\xi + i\eta)$ has large values (on the order of $\log\log\eta$) of either prescribed sign, with ξ tending to 0, and then used a modified form of the Phragmén–Lindelöf principle to show that an equally large value of $-\Im F(i\eta)$ must be attained.

The Phragmén–Lindelöf principle, that the maximum of an analytic function defined on a semi-infinite strip (with suitable growth conditions) must occur on the boundary of the strip, is only an existence result; Skewes strengthens the result to give quantitative bounds on when an approximation to an interior value is attained on the boundary of the strip. In this way he is able to make Littlewood's result explicit (although the details are not included), which was not clearly possible beforehand.

(Estimates of the smallest x such that $\pi(x) > \mathrm{li}(x)$ have since been called "Skewes numbers". For comparison, the best estimates today are about 1.4×10^{316}.)

This article cites [147].

[10] A. E. Ingham, *A note on the distribution of primes*, Acta Arith. **1** (1936), 201–211.

This article follows on [9] to give another proof of an explicit version of "Littlewood's theorem" [5], and to establish the following stronger result: Let Θ be the supremum of the real parts of the zeros of the Riemann zeta function. If Θ is attained, then there exists an absolute constant $A > 1$ such that, for all $x > 1$, the interval (x, Ax) contains a sign change of $\pi(x) - \mathrm{Li}(x)$.

In particular if the Riemann hypothesis is true, then $\Theta = 1/2$ is attained (by every zero) and so the conclusion of the theorem follows. On the other hand, if $\Theta = 1$, then it cannot be attained and no conclusion can be drawn. The author highlights his use of Fejér kernels in the proof, in contrast with the Poisson kernel used in [9].

This article cites [5, 8, 9].

[11] A. Wintner, *On the distribution function of the remainder term of the prime number theorem*, Amer. J. Math. **63** (1941), 233–248. MR0004255

This paper investigates the normalized remainder term $E^{\Psi}(x)$ by establishing the existence of its limiting logarithmic cumulative distribution function. Assuming RH, the author proves that the spectrum of this limiting distribution is unbounded both above and below, which implies the same for $E^{\Psi}(x)$. He also gives an estimate for all of the moments of this limiting distribution, in an effort to determine whether these moments uniquely determine the given distribution. The techniques involved in the proof come primarily from the application of the theory of almost-periodic functions (of both the uniform and Besicovitch varities) to the sum over the non-trivial zeros in the fundamental explicit formula.

This article cites [147–149].

[12] A. E. Ingham, *On two conjectures in the theory of numbers*, Amer. J. Math. **64** (1942), 313–319. MR0006202

In this paper, Ingham probes conjectured bounds for the summatory functions $M(x)$ and $L(x)$. He proves that the truth, for sufficiently large x, of any one of the inequalities $M(x) < Kx^{1/2}$, $M(x) > -Kx^{1/2}$, $L(x) < Kx^{1/2}$, or $L(x) > -Kx^{1/2}$ (where K is a constant) would not only imply RH and the simplicity of the zeros of $\zeta(s)$ (as was "well known"), but also LI. Of this last assumption, the author writes: "It would be easy to relax this hypothesis a little, but there seems no obvious way of replacing it by anything essentially easier to verify." Indeed, he shows that if there are only finitely many rational linear relations among the positive imaginary parts of these zeros, then $M(x)/\sqrt{x}$ and $L(x)/\sqrt{x}$ would be unbounded both above and below, contrary to existing conjectures.

The method of the proof is similar to Littlewood's disproof of the conjecture $\pi(x) < \mathrm{li}(x)$ in [5], including a reliance on trigonometric polynomials involving the zeros of $\zeta(s)$, except that Dirichlet's theorem on homogeneous Diophantine approximation is replaced by Kronecker's theorem on inhomogeneous Diophantine approximation. For the proof, the author establishes two main results, one concerning Laplace transforms of real trigonometric polynomials, and the other establishing the divergence (assuming RH) of the two residue series $\sum_{\gamma>0} 1/\rho\zeta'(\rho)$ and $\sum_{\gamma>0} \zeta(2\rho)/\rho\zeta'(\rho)$.

This article cites [134–136, 138, 141, 143].

[13] C. L. Siegel, *On the zeros of the Dirichlet L-functions*, Annals of Math. **46** (1945), 409–422.

[THIS PAPER WILL EVENTUALLY BE REMOVED]

The author proves a number of results regarding the distribution of zeros of the functions $L(s,\chi)$, for varying character χ as well as varying conductor m.

Let T_0 be a fixed large positive constant. The first result shows that if $1/\log\log m < \delta < 1/2$, then the number of zeros of all $L(s,\chi)$ with χ of given conductor m, in the box $\frac{1}{2}+\delta < \sigma < 1, -T_0 < t < T_0$, is $\ll_{T_0} \phi(m)\cdot\delta^{-1}(\log m)^{-2\delta}$. As m increases we can decrease the chosen δ, and the result then shows that not too many zeros can lie far from the critical line. Choosing $\delta = \frac{1}{2}(\log\log m)(\log\log\log m)^{-1}$ gives that for m large enough, at least one of the functions $L(s,\chi)$ has no zero in the box described.

This contrasts with the next two theorems, which establish that the $L(s,\chi)$ must have more zeros up to height T_0 than counted above. Let $A(T)$ denote the number of zeros of all $L(s,\chi)$, again for fixed conductor m, in the box $0 < \sigma < 1, 0 \le t < T$, for some $T < T_0$. The author shows that

$$\left| A(T) - \frac{\phi(m)}{2\pi} T \log m \right| \ll_{T_0} \phi(m)(\log m)^{2/3}.$$

Finally he proves that if $-T_0 < T_1 < T_2 < T_0$, with $T_2 - T_1 > 4(\log\log\log m)^{-1}$, then each of the functions $L(s,\chi)$ (where χ has conductor m) has a zero in the rectangle $\frac{1}{2} \le \sigma < 1, T_1 < t < T_2$. These results combine to give that every point on the critical line $\sigma = \frac{1}{2}$ is a limit point for the set of zeros of the $L(s,\chi)$ for variable χ and m, and in fact a subset of these functions have zeros that cluster exactly towards all points of the critical line.

[14] S. Skewes, *On the difference $\pi(x) - \mathrm{li}(x)$ (II)*, Proc. London Math. Soc. (3) **5** (1955), 48–70. MR0067145

This article provides an unconditional explicit estimate for a sign change of the difference $\pi(x) - \mathrm{li}(x)$: if we define $X_1 = e^{e^{e^{7.703}}}$ and $X_2 = e^{4X_1^{30}} < e^{e^{e^{e^{7.705}}}} < 10^{10^{10^{10^3}}}$, then the author shows that there exists some $x < X_2$ such that $\pi(x) > \mathrm{li}(x)$. The author divides the proof into two cases, first when RH is "nearly true" and then the contrary case. More specifically, he defines a hypothesis (H) (the "nearly true" case) as follows: *Every zero $\rho = \beta + i\gamma$ for which $|\gamma| < X_1^3$ satisfies $\beta - \frac{1}{2} \le X_1^{-3}\log^{-2} X_1$.*

For the case where (H) holds, the author modifies Ingham's technique from [10], which assumed RH but improved the estimation of $\psi_0(x) - x$ by showing that zeros with γ large relative to x do not contribute meaningfully to the sum. Ultimately the author's argument boils down to estimation of the sum $\sum_{0<\gamma<500} \frac{\sin\gamma\omega}{\gamma}\left(1 - \frac{\gamma}{500}\right)$; Dirichlet's box principle is used again, in conjunction with estimates of the values of the 269 zeros of $\zeta(s)$ with $0 < \gamma < 500$.

For the contrary case, which the author calls (NH), he remarks that it no longer suffices to work first with $\psi(x)$ and then pass to $\pi(x)$ with standard partial summation techniques. Instead, he works directly from the explicit formula for $\Pi_0(x) - \mathrm{li}(x)$, introducing a smoothing factor to amplify the contribution from the hypothesized (H)-violating zero. Throughout, the author uses explicit estimates for sums over nontrivial zeros of $\zeta(s)$, such as $|N(T+h) - N(T)| < \frac{1}{2\pi}(h + 1.77)\log T + 8.7$ for $7.1 < h < \frac{T}{2}$.

This article cites [4, 9, 10, 147].

[15] J. Leech, *Note on the distribution of prime numbers*, J. London Math. Soc. **32** (1957), 56–58. MR0083001

The author uses the EDSAC at Cambridge to compute $\pi(x;4,1)$ and $\pi(x;4,3)$ for x up to 3×10^6. He discovers that $\pi(x;4,1) > \pi(x;4,3)$ at $x = 26{,}861$, for which $\pi(x;4,1) = 1{,}473$ and $\pi(x;4,3) = 1{,}472$. The other values of x above 26,863 for which $\pi(x;4,1) > \pi(x;4,3)$ are between 616,000 and 634,000; the greatest difference found is at $x = 623{,}681$, for which $\pi(x;4,1) = 25{,}444$ and $\pi(x;4,3) = 25{,}436$. The author notes that $\pi_i(x) = 2\pi(x;4,1) + \pi(\sqrt{x};4,3) + 1$, the number of Gaussian primes with norm at most x (up to associates, and for $x \ge 2$), is consequently large near this latter range as well; the most extreme value found is at $x = 617{,}537$, for which $\pi_i(x) = 50{,}509 \approx \mathrm{li}(x) + 19.5$.

When examining the explicit formula for $\pi(x;4,3,1)$ at $x = 620{,}000$, the author found that the first 20 pairs of zeros of $L(s,\chi_{-4})$, whose imaginary parts ranged from ± 6.020948 to ± 49.723129, included 16 pairs that give negative contributions to the explicit formula, while subsequent zeros gave more or less random contributions.

This article cites [9, 147].

[16] D. Shanks, *Quadratic residues and the distribution of primes*, Math. Tables Aids Comput. **13** (1959), 272–284. MR0108470

In this paper, the author investigates Chebyshev's assertion that there are more primes of the form $4m-1$ than of the form $4m+1$. Define $\tau(n) = \pi(n;4,3,1)\sqrt{n}/\pi(n)$ (which is asymptotically equivalent to $\pi(n;4,3,1)\log n/\sqrt{n}$ but is easier to manipulate numerically). Upon computing $\pi(n;4,3,1)$ for values of n up to 3 million, he analyzes the values $\tau(1000k)$ for $1 \le k \le 2000$, noting that their histogram is "roughly normal with a mean of (nearly) 1". The author conjectures that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \le x} \tau(n) = \lim_{x\to\infty} \frac{1}{x} \sum_{n \le x} \frac{\pi(x;4,3,1)\sqrt{n}}{\pi(n)} = 1,$$

and notes that weaker versions of the conjecture—namely, that the above limit holds under GRH for $L(s,\chi_{-4})$, or that the above limit either equals 1 or fails to exist—are also open. (He can show, under GRH, that the mean value inside the limit is positive and bounded away from 0 for sufficiently large x.)

Next, the author discusses the distribution of primes in the residue classes modulo 8, 10, and 12. Both from examining the collected data and from combinatorial reasoning involving the multiplicative groups of those moduli, he concludes that the quadratic residues are the ones with a smaller number of primes (on average). For the specific modulus 4, he outlines an argument, based on combinatorial reasoning with the quantities $\#\{n \le x \colon \Omega(n) = a,\ n \equiv \pm 1 \pmod 4\}$, that shows that the mean value of $\tau(n)$ should be 1. Indeed, he remarks that the generalization of this mean value to the integers with a prime factors (counted with multiplicity) predicts that it is the residue class $(-1)^a \pmod 4$ that should have more such integers; in other words, the bias switches according to the parity of the number of prime factors. A related remark is that there is a bias towards integers for which $\lambda(n)\chi_{-4}(n)$ equals 1 over those for which it equals -1.

This paper ends with a discussion of how similar arguments to those laid out in this paper could be used to analyze the relationship between $\pi(x)$ and $\mathrm{li}(x)$.

This article cites [1–3, 5, 9, 15, 155].

[17] S. Knapowski, *On sign-changes in the remainder-term in the prime-number formula*, J. London Math. Soc. **36** (1961), 451–460. MR0133309

The author shows that if $\rho_0 = \beta_0 + i\gamma_0$ is any zero of $\zeta(s)$, then for T sufficiently large in terms of γ_0,

$$\Delta^\psi(t) = \Omega_\pm\biggl(T^{\beta_0} \exp\biggl(\frac{-15\log T}{\sqrt{\log\log T}}\biggr)\biggr),$$

and the same for $\Delta^\Pi(t)$ (which implies an Ω_--result, though not an Ω_+-result, for $\Delta^\pi(t)$). The author also observes that these large oscillations in $\Delta^\psi(t)$ cannot occur when $t < T^{1/2-\varepsilon}$, which implies that $\liminf_{T\to\infty} W^\psi(T)/\log\log T \ge 1/\log 2$.

This article cites [10, 14, 153].

[18] ———, *On sign-changes of the difference* $\pi(x) - \operatorname{li} x$, Acta Arith. **7** (1961/1962), 107–119. MR0133308

This article is concerned with explicit lower bounds for the number $W(T)$ of sign changes of the function $\operatorname{li}(x) - \pi(x)$. Previously, Ingham [10] had shown, assuming SA, that $\liminf_{T\to\infty} W(T)/\log T > 0$, while the author [17] had proved unconditionally that $\liminf_{T\to\infty} W(T)/\log\log T > 0$. Skewes [14] famously found that $W\bigl(e^{e^{e^{e^{7.705}}}}\bigr) \ge 1$.

In this paper, the author shows unconditionally that $W(T) \ge e^{-35} \log\log\log\log T$ for $T \ge e^{e^{e^{e^{35}}}}$ (the author did not try to optimize these constants). Similar to [14], the proof is divided into two cases, first when RH is "nearly true" and then the contrary case. More specifically, setting $X = \sqrt[7]{\log\log T}$, the author defines a hypothesis (C) (the "nearly true" case) as follows: *Every zero* $\rho = \beta + i\gamma$ *for which* $|\gamma| \le X^3$ *satisfies* $\beta - \frac12 \le 2/(3X^3\log X)$. The author then proves the inequality above first assuming (C) and then again assuming its negation (NC).

This article cites [4, 14, 17, 147].

[19] S. Knapowski and P. Turán, *Comparative prime-number theory. I. Introduction*, Acta Math. Acad. Sci. Hungar. **13** (1962), 299–314. MR0146156

The authors start by introducing ten problems of interest in "comparative prime-number theory" to the modulus k, the first seven concerning the sign changes and extreme values of $\pi(x; k, \ell_1, \ell_2)$ and the natural density of the solutions to $\pi(x; k, \ell_1, \ell_2) > 0$. The eighth problem, which the authors call the "race-problem of Shanks–Rényi" (which is perhaps the first time Rényi's name was linked to comparative prime number theory) is whether there are arbitrarily large solutions x to $\pi(x; k, \ell_1) < \cdots < \pi(x; k, \ell_{\phi(k)})$; the last two problems concern the simultaneous inequalities $\pi(x; k, \ell_j) > \frac{1}{\phi(k)} \operatorname{li}(x)$.

The authors allude to variants of these ten problems generated by replacing $\pi(x; k, \ell)$ by $\pi_e(x; k, \ell)$ (and, where needed, $\operatorname{Li}(x)$ by $\int_2^\infty \frac{e^{-t/x}}{\log t}\,dt$), and further vary these problems by replacing π with ψ or Π. They are aware that such problems could be further varied ("… the analogous problems concerning the distribution of primes in binary quadratic forms with fixed discriminant or of the prime ideals of a fixed field in various idealclasses").

In Section 4, the authors discuss how some of the problems involving $\psi(x; k, \ell)$ can be conditionally solved using Laudau's theorem (and hence unconditionally for moduli dividing 24). In Section 5, the authors discuss the results they have so far concerning the problems involving $\pi(x; k, \ell)$, and in Section 8, they briefly discuss what is known about prime number races modulo 4. Throughout the rest of this paper, the authors introduce the results that will be proved in the next seven papers of the series.

This article cites [1, 2, 5, 8, 10, 14, 16].

[20] ———, *Comparative prime-number theory. II. Comparison of the progressions* $\equiv 1 \bmod k$ *and* $\equiv l \bmod k$, $l \not\equiv 1 \bmod k$, Acta Math. Acad. Sci. Hungar. **13** (1962), 315–342. MR0146157

This article focuses on the race between the residue class 1 (mod k) and other residue classes $\ell \not\equiv 1 \pmod{k}$, for a fixed modulus k for which HC holds.

Fix a character $\chi \pmod k$ such that $\chi(\ell) \neq 1$, and let $\rho_0 = \beta_0 + i\gamma_0$ be a zero of $L(s,\chi)$. The authors prove that for T large enough,

$$\max_{T^{1/3} \le x \le T} \psi(x;k,1,\ell) > T^{\beta_0} \exp\left(-41 \frac{\log T \log\log\log T}{\log\log T}\right)$$
$$\max_{T^{1/3} \le x \le T} \Pi(x;k,1,\ell) > T^{\beta_0} \exp\left(-41 \frac{\log T \log\log\log T}{\log\log T}\right)$$

and symmetric results for the minimum. As one might expect, the oscillations obtained for $\pi(x;k,1,\ell)$ are worse, but the authors do prove

$$\max_{\exp(\log_3^{1/130} T) \le x \le T} \left(\frac{\log x}{\sqrt{x}}\right) \pi(x;k,1,\ell) > \frac{1}{100} \log\log\log\log\log T$$

and the symmetric result for the minimum. Each of these results yields lower bounds on the corresponding number of sign changes, although for $\pi(x;k,1,\ell)$ the bound is very low—improving this bound is addressed directly in [21]. The authors' methods can also compaire primes congruent to 1 (mod k) to the average number of primes in other residue classes (mod k):

$$\max_{\exp(\log_3^{1/130} T) \le x \le T} \left(\frac{\log x}{\sqrt{x}}\right) \left(\pi(x;k,1) - \frac{1}{\phi(k)-1} \sum_{\substack{(\ell,k)=1 \\ \ell \neq 1}} \pi(x;k,\ell)\right) > \frac{1}{100} \log_5 T$$

and the symmetric result for the minimum.

The proofs follow from the application of Turán's method for bounding exponential sums. Siegel's theorem on the existence of zeros in certain rectangles, coupled with the verification of HC for certain moduli up to 24, give for these moduli the first unconditional results about the size of the fluctuations of the above functions and the number of their sign changes.

This article cites [4, 5, 10, 14, 17].

[21] ______, *Comparative prime-number theory. III. Continuation of the study of comparison of the progressions* $\equiv 1 \bmod k$ *and* $\equiv l \bmod k$, Acta Math. Acad. Sci. Hungar. **13** (1962), 343–364. MR0146158

The authors continue their comparison of $\pi(x;k,\ell)$ and $\pi(x;k,1)$, where $\ell \not\equiv 1 \pmod k$ and $(\ell,k)=1$, assuming HC for the modulus k. They show that $W_{k;\ell,1}(T) > k^{-c} \log\log\log\log T$ for sufficiently large T, where c is an absolute effective constant. The proof technique involves Dirichlet's box principle and bounds obtained on $\Pi(x;k,\ell,1)$ in [20].

When ℓ is a quadratic residue (mod k), they also show that if $\rho_0 = \beta_0 + i\gamma_0$ is a zero of $L(s,\chi)$ for some character χ such that $\chi(\ell) \neq 1$, then for sufficiently large T,

$$\max_{T^{1/3} \le x \le T} \pi(x;k,\ell,1) > T^{\beta_0} \exp\left(-42 \frac{\log T \log\log\log T}{\log\log T}\right),$$

and a similar statement holds for the minimum; consequently, for such k and ℓ, the inequality $W_{k;\ell,1}(T) > \frac{1}{\log 3} \log\log T + O(1)$ holds for sufficiently large T. The proof involves Turán's method for bounds on exponential sums.

The authors remark that both theorems hold as well for $W\big(\phi(k)\pi(x;k,1), \pi(x); T\big)$ and $W\big(\phi(k)\pi(x;k,1), \mathrm{Li}(x); T\big)$.

This article cites [5, 9, 10, 13, 19, 20].

[22] ______, *Comparative prime-number theory. IV. Paradigma to the general case,* $k = 8$ *and* 5, Acta Math. Acad. Sci. Hungar. **14** (1963), 31–42. MR0146159

The authors apply techniques from earlier in this series of papers to the modulus $k = 8$, when $\ell_1, \ell_2 \in \{3,5,7\}$ are distinct quadratic nonresidues. They show that

$$\max_{T^{1/3} \le x \le T} \pi(x;8,\ell_1,\ell_2) > \sqrt{T} \exp\left(-23 \frac{\log T \log\log\log T}{\log\log T}\right),$$

and similarly for $\Pi(x;8,\ell_1,\ell_2)$ and $\psi(x;8,\ell_1,\ell_2)$. Since ℓ_1 and ℓ_2 can be interchanged, this result implies the inequality $W_{8;\ell_1,\ell_2}(T) > \frac{1}{\log 3} \log\log T + O(1)$, and similarly for $W^{\Pi}_{8;\ell_1,\ell_2}(T)$ and $W^{\psi}_{8;\ell_1,\ell_2}(T)$.

In the third section, the authors remark on the modulus $k = 5$. The case $(\ell_1, \ell_2) = (\ell, 1)$ has already been handled earlier in this series; the authors mention that the case $(\ell_1, \ell_2) = (2, 3)$, where both are quadratic nonresidues, can be handled in a similar way to the $k = 8$ cases treated in this paper. The remaining cases have $\ell_1 = 4$, a quadratic residue not equal to 1 (mod 5), and $\ell_2 \in \{2, 3\}$, and these cases yield "an unpleasant (or pleasant?) surprise": the authors cannot establish sign changes for $\pi(x; 5, 4, 2)$ and $\pi(x; 5, 4, 3)$ even assuming GRH (although the methods do work for the Π and ψ versions, a situation the authors discuss further in the later papers of this series).

This article cites [19–21].

[23] ———, *Comparative prime-number theory. V. Some theorems concerning the general case*, Acta Math. Acad. Sci. Hungar. **14** (1963), 43–63. MR0146160

Under the assumption of a "finite Riemann–Piltz conjecture" GRH$(H(k), A(k))$, the authors establish the following result for any distinct reduced residues $\ell_1, \ell_2 \pmod k$ and for T sufficiently large (explicitly quantified in the paper):

$$\max_{T^{1/3} \le x \le T} \Pi(x; k, \ell_1, \ell_2) > \sqrt{T} \exp\left(-44 \frac{\log T \log\log\log T}{\log\log T}\right),$$

and the same with Π replaced by ψ. A central element to their proof is the estimation of the integral

$$\begin{aligned} J(T) &= -\frac{1}{2\pi i} \int_2 \left(\frac{e^{y_1 s}}{s}\right)^v \frac{(\omega_0 L_1^{v_0})^s}{s^{v_0+1}} \cdot \frac{1}{\phi(k)} \Bigg\{ \sum_{\substack{\chi \pmod k \\ \chi(\ell_1) \ne \chi(\ell_2)}} (\overline{\chi}(\ell_1) - \overline{\chi}(\ell_2)) \frac{L'}{L}(s, \chi) \Bigg\} \, ds \\ &= \frac{1}{(v+v_0)!} \int_1^{Y_1} \Pi(x; k, \ell_1, \ell_2) \frac{d}{dx} \left(\left(\log \frac{Y_1}{x} \right)^{v+v_0} \log x \right) dx. \end{aligned}$$

Their approach involves an application of Turán's method similar to what appears in the previous papers of the series. The authors note that their main theorem gives similar bounds on $\Pi(x; k, \ell) - \mathrm{Li}(x)/\phi(k)$ and $\psi(x; k, \ell) - x/\phi(k)$. In particular, this result implies the lower bounds $W^{\Pi}_{k;\ell_1,\ell_2}(T) > \frac{1}{\log 3} \log\log T + O(1)$ and $W^{\psi}_{k;\ell_1,\ell_2}(T) > \frac{1}{\log 3} \log\log T + O(1)$.

This article cites [13, 19–22, 162].

[24] ———, *Comparative prime-number theory. VI. Continuation of the general case.*, Acta Math. Acad. Sci. Hungar. **14** (1963), 65–78. MR0146161

Under the assumption of a "finite Riemann–Piltz conjecture" GRH$(H(k), A(k))$, the authors establish the following result in the case that ℓ_1 and ℓ_2 are either both quadratic residues or both quadratic nonresidues (mod k): when T is sufficiently large in terms of k (the authors give an explicit lower bound), the inequalities

$$\begin{aligned} \max_{T^{1/3} \le x \le T} \pi(x, k, \ell_1, \ell_2) &> \sqrt{T} \exp\left(-44 \frac{\log T \log\log\log T}{\log\log T}\right) \\ \min_{T^{1/3} \le x \le T} \pi(x, k, \ell_1, \ell_2) &< -\sqrt{T} \exp\left(-44 \frac{\log T \log\log\log T}{\log\log T}\right) \end{aligned}$$

hold. As usual, this result implies the lower bound $W_{k;\ell_1,\ell_2}(T) > \frac{1}{\log 3} \log\log T + O(1)$. The authors rely on multiple lemmas from their use of Turán's method in previous papers of this series.

This article cites [20, 21, 23].

[25] ———, *Comparative Prime-Number Theory VII*, Acta Math. Acad. Sci. Hung. **14** (1963), 241–250.

This article gives a general conditional proof that $\psi(x; k, \ell_1, \ell_2)$ changes sign infinitely often. The authors show, assuming HC$(A(k))$ for the modulus k for some constant $0 < A_k \le 1$, that there exists a positive constant c such that $\psi(x; k, \ell_1, \ell_2)$ changes sign in every interval of the form $\omega \le x \le \exp(2\sqrt{\omega})$ as long as

$$\omega \ge \max\left\{e^{k^c}, e^{2/A(k)^3}\right\}.$$

This result immediately implies results for the first sign change of $\psi(x; k, \ell_1, \ell_2)$ and for its number of sign changes.

This article cites [19–21, 23, 24].

[26] ______, *Comparative Prime-Number Theory VIII*, Acta Math. Acad. Sci. Hung. **14** (1963), 251–268.

Hardy–Littlewood and Landau had already shown that the assertion $\lim_{x\to\infty} \pi_e(x;4,1,3) = -\infty$ is equivalent to GRH for $L(s,\chi_{-4})$. In this article the authors obtain an analogous equivalence concerning the races between 1 and a nonsquare (mod 8): slightly modifying the arguments for the (mod 4) case, they show that the assertion $\lim_{x\to\infty} \theta_e(x;8,1,\ell) = -\infty$ for all $\ell \not\equiv 1 \pmod 8$ is equivalent to GRH for the three nonprincipal Dirichlet L-functions (mod 8), and the same for the assertion $\lim_{x\to\infty} \pi_e(x;8,1,\ell) = -\infty$.

They further show that the race between two nonsquares switches infinitely often—more precisely, for $\ell_1 \not\equiv \ell_2 \not\equiv 1 \pmod 8$, they unconditionally show that when T is large enough,

$$\max_{T^{1/3}\le x\le T} \theta_e(x;8,\ell_1,\ell_2) > \sqrt{T}\exp\left(-22\frac{\log T \log\log\log T}{\log\log T}\right).$$

They indicate that this result is "deeper", and in particular that they cannot yet replace θ_e with π_e in this result.

The proofs rely on Turán's method, as well as some explicit numerical data for the low-lying zeros of the L-functions (mod 8).

This article cites [5–7].

[27] I. Kátai, *Eine Bemerkung zur "Comparative prime-number theory I–VIII" von S. Knapowski und P. Turán*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **7** (1964), 33–40 (German). MR0176967

Let ℓ_1 and ℓ_2 be distinct reduced residues (mod k). Assuming HC, the author proves that

$$\limsup_{x\to\infty} \frac{\psi(x,k,\ell_1,\ell_2)}{\sqrt{x}} > 0$$

(and hence the corresponding statement for lim inf). When ℓ_1 and ℓ_2 are either both quadratic residues or both quadratic nonresidues, it follows that

$$\limsup_{x\to\infty} \frac{\pi(x,k,\ell_1,\ell_2)}{\sqrt{x}/\log x} > 0,$$

(and the corresponding statement for lim inf). The proof uses an idea of Littlewood, namely to estimate the iterated integrals $\Delta_n(x) = \int_2^x \Delta_{n-1}(u)\,du$ where $\Delta_0(x) = \psi(x;k,\ell_1,\ell_2) + O(\log x)$ is the explicit sum over zeros of Dirichlet L-functions (mod k).

Assuming GRH, the author can make the above statements quantitative and localized to intervals of the form (x_0, ax_0), thus obtaining the lower bounds $W^{\psi}_{k;\ell_1,\ell_2}(T) \gg \log x$ and (under the same assumption on ℓ_1 and ℓ_2) the same estimate for $W^{\psi}_{k;\ell_1,\ell_2}(T) \gg \log x$.

This article cites [3, 19–26, 145].

[28] S. Knapowski and P. Turán, *Further developments in the comparative prime-number theory I*, Acta. Arith. **9** (1964), 23–40. MR0162771

The first two sections offer a short summary of comparative prime number theory up to 1964. The authors classify the subject into 48 separate problems over 12 categories (and additional variants on these), which is of interest to those interested in the history of the field. They then move on to prove results about "strongly localized accumulation problems".

Most generally, assuming HC for the modulus k, they show that when T is sufficiently large in terms of k, then for any $(\ell,k)=1$ with $\ell \not\equiv 1 \pmod k$,

$$\max_I \psi\big(I;k,\ell,1\big) > \sqrt{T}e^{-\log^{11/12} T} \quad\text{and}\quad \min_I \psi\big(I;k,\ell,1\big) < -\sqrt{T}e^{-\log^{11/12} T},$$

where the maximum and minimum are taken over all subintervals I of $\big[Te^{-\log^{11/12} T}, T\big]$. The central argument involves the evaluation of the integral

$$\frac{1}{2\pi i}\int e^{As}\left(\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r \left(\frac{1}{\varphi(k)}\sum_{\chi \pmod k} \overline{\chi}(\ell)\frac{L'}{L}(s,\chi)\right) ds$$

for positive constants A and B.

This article cites [1, 5–7, 13, 17, 19–26, 156].

[29] ———, *Further developments in the comparative prime-number theory II*, Acta. Arith. **10** (1964), 293–313.

The authors establish several theorems, all assuming HC(E_k) with $E_k \ll \sqrt{\log k}/k$; most of their results concern the function $\theta_l(x, r; k, \ell_1, \ell_2)$ where $r = r(x, k)$ satisfies $\frac{\log k}{E_k} \ll r \le \log x$. Their most general result (Theorem VI) is that for any quadratic nonresidue $\ell_1 \pmod{k}$ and quadratic residue $\ell_2 \pmod{k}$, if $L(s, \chi)$ satisfies GRH for all characters $\chi \pmod{k}$ such that $\chi(\ell_1) \ne \chi(\ell_2)$, then $\theta_l(x, r; k, \ell_1, \ell_2) \gg \sqrt{x}$ for x sufficiently large.

They also establish the following result. Let ℓ be a quadratic nonresidue $\pmod{k}$, and suppose that there exists a character $\chi \pmod{k}$ with $\chi(\ell) \ne 1$ such that $L(s, \chi)$ has a zero $\rho_0 = \beta_0 + i\gamma_0$ with $\beta_0 > \frac{1}{2}$. Then for all T (sufficiently large in terms of k and ρ_0), there exist subintervals $I_\pm$ of $\left[Te^{-5\log^{20/21} T}, Te^{5\log^{20/21} T}\right]$ such that $\pi(I_+; k, \ell, 1) > T^{\beta_0} \exp\left(-(2+\gamma_0^2)(\log T)^{5/7}\right)$ and $\pi(I_-; k, \ell, 1) < -T^{\beta_0} \exp\left(-(2+\gamma_0^2)(\log T)^{5/7}\right)$. They deduce this theorem from an analogous theorem involving $\theta_l(x, r; k, \ell, 1)$, which serves as a sort of inverse to Theorem VI.

Together, these results imply, given a quadratic nonresidue $\ell \pmod{k}$, that the limit $\lim_{x\to\infty} \theta_l(x, r; k, \ell, 1) = +\infty$ holds if and only if $L(s, \chi)$ satisfies GRH for all characters $\chi \pmod{k}$ with $\chi(\ell) \ne 1$. It follows that $\lim_{x\to\infty} \theta_l(x, r; k, \ell, 1) = +\infty$ holds for all quadratic nonresidues $\pmod{k}$ if and only if $L(s, \chi)$ satisfies GRH for every nonprincipal character $\chi \pmod{k}$.

The authors also make some remarks about races between residue classes to different moduli, showing for example how the race between $\pi(x; 3, 1)$ and $\pi(x; 4, 1)$ reduces to a race between residue classes modulo 12, to which their results apply.

This article cites [1, 5, 6, 17, 20, 22, 26, 156].

[30] ———, *Further developments in the comparative prime-number theory III*, Acta. Arith. **11** (1965), 115–127.

This paper concerns the weighted function $\theta_l(x, r; k, \ell, 1)$, for a modulus k satisfying HC and a quadratic residue $\ell \not\equiv 1 \pmod{k}$. Let β_0 be the real part of any zero of an $L(s, \chi)$ where $\chi \pmod{k}$ a character such that $\chi(\ell) \ne 1$. The authors exhibit extreme values of $\theta_l(x, r; k, \ell, 1)$ where x is near T and r is near $(\log T)^{2/3}$; more precisely, there exists a positive constant c such that for T sufficiently large, there exist $x_1, x_2 \in (Te^{-(\log T)^{5/6}}, Te^{(\log T)^{11/15}})$ such that for suitable $r_1, r_2 \in [(2\log T)^{2/3}, (2\log T)^{2/3} + (2\log T)^{2/5}]$,

$$\theta_l(x_1, r_1; k, \ell, 1) > T^{\beta_0} e^{-c(\log T)^{5/6}} \quad \text{and} \quad \theta_l(x_2, r_2; k, \ell, 1) < -T^{\beta_0} e^{-c(\log T)^{5/6}}.$$

The authors then state that using the methods of their prior paper [29], it follows that for T sufficiently large, there exist closed subintervals $I, J \subseteq [Te^{-(\log T)^{6/7}}, Te^{(\log T)^{6/7}}]$ such that one has the "strongly localized accumulations"

$$\pi(I; k, \ell, 1) > \sqrt{T} e^{-c(\log T)^{5/6}} \quad \text{and} \quad \pi(J; k, \ell, 1) < -\sqrt{T} e^{-c(\log T)^{5/6}}.$$

This article cites [1, 5–7, 13, 20, 26, 29, 156].

[31] ———, *Further developments in the comparative prime-number theory IV*, Acta Arith. **11** (1965), 147–161. MR0182616

Let ℓ_1 and ℓ_2 be quadratic non-residues modulo a sufficiently large modulus k. Let η be sufficiently small in terms of k, and suppose that the Dirichlet L-functions $\pmod{k}$ satisfy GRH($2/\sqrt{\eta}, E_k$) for suitable E_k. Then, when T is sufficiently large in terms of k and η, there exist x_+ and x_- in the interval $[T^{1-\sqrt{\eta}}, Te^{\log^{3/4} T}]$, and η_1 and η_2 in the interval $[2\eta \log T, 2\eta \log T + \sqrt{\log T}]$, such that $\theta_l(x_+, v_+; k, \ell_1, \ell_2) > T^{1/2 - 4\sqrt{\eta}}$ and $\theta_l(x_-, v_-; k, \ell_1, \ell_2) < -T^{1/2-4\sqrt{\eta}}$. Furthermore, under the same assumptions, there exist subintervals I_+, I_- of $[T^{1-4\sqrt{\eta}}, T^{1+4\sqrt{\eta}}]$ such that $\pi(I_+; k, \ell_1, \ell_2) > T^{1/2-5\sqrt{\eta}}$ and $\pi(I_-; k, \ell_1, \ell_2) < -T^{1/2-5\sqrt{\eta}}$.

This article cites [13, 21, 23, 29, 156].

[32] ———, *Further developments in the comparative prime-number theory V*, Acta Arith. **11** (1965), 193–202. MR0182616

This short paper is distinct among the second series by Knapowski and Turan, in that, rather than make use of Turan's "one-sided" methods, it uses a different, "two-sided" theorem to obtain its results: If $m > 0$ and $z_1, \dots, z_n \in \mathbb{C}$ non-decreasing in absolute value with $|z_1| = 1$, then for any $b_1, \dots, b_n \in \mathbb{C}$, there exists an integer ν such that $m \le \nu \le m+n$ and

$$\left| \sum_{j=1}^{n} b_j z_j \right| \ge \frac{1}{2n} \left(\frac{n}{8e(m+n)} \right)^n \min_{1\le k\le n} \left| \sum_{j=1}^{k} b_j \right|.$$

In addition to the use of the "two-sided" theorem above, the authors use a modified idea attributed to Kreisel involving a sequence of integrals.

Their main result of their paper is a single theorem, for residues $\ell \not\equiv 1 \pmod{k}$ for sufficiently large moduli k, under the assumption that there exists $0 < \delta < \frac{1}{10}$ such that no function $L(s,\chi)$ with $\chi(\ell) \ne 1$ vanishes in the closed disk $|s-1| \le \frac{1}{2} + 4\delta$. (This assumption is stronger than HC($2\sqrt{\delta}$) but weaker than HC($\frac{1}{2}+4\delta$).) For any sufficiently large T, the interval $I = [T, e^{(\log T)^2(\log\log T)^3}]$ contains x_1, x_2 such that

$$\psi(x_1; k, 1, \ell) \ge x_1^{1/2-4\delta} \quad \text{and} \quad \psi(x_2; k, 1, \ell) \le -x_1^{1/2-4\delta}.$$

The authors compare their result to [20, Theorem 1.1], which uses more conventional methods and yields a more localized sign change.

This paper cites [18, 20, 164].

[33] ———, *On an assertion of Čebyšev*, J. Analyse Math **14** (1965), 267–274.

The authors begin by remarking on some variants of the result of Hardy–Littlewood–Landau [5–7] that Chebyshev's assertion, namely that $\lim_{x\to\infty} \pi_e(x; 4, 1, 3) = -\infty$, is equivalent to GRH for $L(s, \chi_{-4})$. The same methods would show the "Abelian preponderance-relations" that $\lim_{x\to\infty} \pi_e(x; 3, 1, 2) = -\infty$ if and only if GRH holds for $L(s, \chi_{-3})$, while $\lim_{x\to\infty} \pi_e(x; 8, 1, \ell) = -\infty$ for all $\ell \in \{3, 5, 7\}$ if and only if GRH holds for all nonprincipal Dirichlet L-functions (mod 8), and ("mutatis mutandis") $\lim_{x\to\infty} \pi_e(x; 12, 1, \ell) = -\infty$ for all $\ell \in \{5, 7, 11\}$ if and only if GRH holds for all nonprincipal Dirichlet L-functions (mod 12). All of these results, they point out, hold with π_e replaced by θ_e.

For the modulus $k = 8$, in the case where $\ell_1, \ell_2 \in \{3, 5, 7\}$ are distinct quadratic nonresidues, the authors had shown [26] that

$$\max_{T^{1/3}\le x\le T} \theta_e(x; 8, \ell_1, \ell_2) > \sqrt{T} \exp\left(-22 \frac{\log T \log\log\log T}{\log\log T} \right);$$

however, they point out that the method failed to yield the analogous result for the "properly Čebyšev" function π_e. In this article, the authors do establish analogous large oscillations (without identifying the signs of those oscillations) in the form

$$\max_{T^{1/3}\le x\le T} |\pi_e(x; 8, \ell_1, \ell_2)| \ge \sqrt{T} \exp\left(-23 \frac{\log T \log\log\log T}{\log\log T} \right)$$

for any distinct reduced residues $\ell_1, \ell_2 \pmod{8}$, as well as the analogous statement for $\pi_e(x; 4, 1, 3)$. The additional technical tool is a result (then unpublished) of Szegö that derives estimates for $\sum_{j=1}^{n} b_j e^{-jy} \log j$ from estimates for $\sum_{j=1}^{n} b_j e^{-jy}$.

This article cites [1, 5–7, 26].

[34] ———, *Further developments in the comparative prime-number theory VI*, Acta Arith. **12** (1966), 85–96.

The authors consider a "modified Abelian means" race between two quadratic residues $\ell_1, \ell_2 \pmod{k}$, under the assumption $GRH(\frac{3}{\sqrt{\eta}}, E_k)$ for suitable constants η and E_k. Their main result is that there exist constants $x \in [T^{1-\sqrt{\eta}}, T \log T]$ and $\nu \sim 2\eta \log T$ such that

$$\theta_l(x, \nu; k, \ell_1, \ell_2) > T^{1/2-2\sqrt{\eta}},$$

(and thus the symmetric result for a large negative value). A corollary is the existence of an interval $I \subset [T^{1-4\sqrt{\eta}}, T^{1+4\sqrt{\eta}}]$ such that

$$\pi(I; k, \ell_1, \ell_2) > T^{1/2-3\sqrt{\eta}}.$$

The authors obtained similar bounds for two quadratic nonresidues in [31], but emphasize they have not been able to extend the results to races where $\ell_1 \not\equiv 1 \pmod k$ is a quadratic residue and ℓ_2 is quadratic nonresidue. They employ Turan's method for exponential sums.

This paper cites [13, 21, 29–31].

[35] I. Kátai, *On investigations in the comparative prime number theory*, Acta Math. Acad. Sci. Hungar **18** (1967), 379–391.

The author establishes an oscillation theorem of Landau type for Dirichlet integrals with a nonreal pole of arbitrary mulitplicity, with the additional feature that the oscillations can be localized to explicit intervals of the form $[T, T^K]$. From this, he deduces many unconditional number-theoretical results. For example, all of the following oscillations can be found in all sufficiently large intervals of the form $[T, T^{7+4\sqrt{3}}]$:

- $M(x) = \Omega_\pm(\sqrt{x})$, and the same for $M(x, \chi_{-4})$ and $M(x; 4, 1)$ and $M(x; 4, 3)$ and $M_e(x)$
- $M_r(x) = \Omega_\pm(1/\sqrt{x})$, and the same for $\sum_{n=1}^\infty \frac{\mu(n)}{n} e^{-(x/n)^2}$
- for any $k \geq 2$, the number of k-free integers up to x is $x/\zeta(k) + \Omega_\pm(x^{1/2k})$, and similarly for the sum of $e^{-n/x}$ over all k-free numbers
- $\psi(x; 8, \ell_1, \ell_2) = \Omega_\pm(x^{1/2})$ and, if $\ell_1, \ell_2 \not\equiv 1 \pmod 8$, then $\pi(x; 8, \ell_1, \ell_2) = \Omega_\pm(x^{1/2})/\log x$

Furthermore, by considering separately the cases where RH or GRH is true or false, the author finds all the following oscillations in intervals of the form $[T, T^{1+\varepsilon}]$ for any fixed $\varepsilon > 0$:

- $M(x) = \Omega_\pm(x^{\Theta-\varepsilon})$, and similarly for the other functions in the previous list
- under HC, $\psi(x; k, \ell_1, \ell_2) = \Omega_\pm(x^{\Theta(k)-\varepsilon})$, and the same for $\psi_e(x; k, \ell_1, \ell_2)$ (indeed, a hypothesis slightly weaker than HC is required, in that real zeros of different L-functions could cancel each other out)

These last theorems imply qualitative improvements on the number of sign changes of their respective functions: for example, $W(M, T)/\log\log T \to \infty$ and, under HC, $W^\psi_{k,\ell_1,\ell_2}(T)/\log\log T \to \infty$.

This article cites [3, 5, 19–26].

[36] H. G. Diamond, *Two oscillation theorems*, The theory of arithmetic functions (Proc. Conf., Western Michigan Univ., Kalamazoo, Mich., 1971), Springer, Berlin, 1972, pp. 113–118. Lecture Notes in Math., Vol. 251. MR0332684

The author presents two variants of oscillation theorems analogous to those of Ingham in [12]. Let $F(s) = \int_0^\infty e^{-su} f(u)\,du$ denote the Laplace transform of the measurable function $f\colon [0,\infty) \to \mathbb{R}$. We suppose that the integral defining $F(s)$ converges for $\Re(s) > 0$, and that $F(s)$ can be continued as a meromorphic function to a neighborhood of the imaginary axis; suppose further that all the poles of $F(s)$ on the imaginary axis are simple. Let T be the set of positive real numbers t such that it is a pole of $F(s)$, and let a_t be the residue of $F(s)$ at $s = it$; furthermore, let a_0 be the residue (possibly 0) of $F(s)$ at $s = 0$.

The author defines a subset $W \subset T$ to be "weakly independent of order N" if the only way to find integers $|n_t| \leq N$ $(t \in W)$ such that $\sum_{t\in W} n_t t \in T$ is to choose one n_t equal to 1 and the rest equal to 0. Given such a weakly independent subset $W \subset T$ of order N, the author proves that

$$\lim_{x\to\infty} \operatorname*{ess\,sup}_{u\geq x} f(u) \geq a_0 + \frac{2N}{N+1} \sum_{j\in J} |a_j|$$
$$\lim_{x\to\infty} \operatorname*{ess\,inf}_{u\geq x} f(u) \leq a_0 - \frac{2N}{N+1} \sum_{j\in J} |a_j|$$

(where these essential supremum and infemum denote the supremum/infemum when we may ignore a set of inputs of measure 0); equivalently, if $\frac{2N}{N+1}\sum_{j\in J} |a_j| > |a_0|$ then $f(x)$ has arbitrarily large sign changes. (The author gives a slight strengthening of this theorem as well.)

This article cites [12, 159, 173, 177, 186].

[37] H. Stark, *A problem in comparative prime number theory*, Acta Arith. **18** (1971), 311–320.

The author states a Tauberian theorem of Landau type (with a proof sketch and a reference to an unpublished paper of the author) and uses it to show that

$$\limsup_{x\to\infty} E^\pi(x;q,a,b) \ge c_q(b) - c_k(a) + \sum_{\substack{\chi \pmod q \\ \chi\ne\chi_0}} \sum_{|\gamma|<T} \frac{\overline{\chi}(b)}{\rho}\left(1-\frac{\gamma}{T}\right)e^{(\rho-1/2)u} - \sum_{\substack{\chi \pmod k \\ \chi\ne\chi_0}} \sum_{|\gamma|<T} \frac{\overline{\chi}(a)}{\rho}\left(1-\frac{\gamma}{T}\right)e^{(\rho-1/2)u}$$

for any $T>0$ and any $u\in\mathbb{R}$, under the assumption of GRH(0) for nonprincipal L-functions modulo k and q. In particular, the lim sup is positive if either a is a nonsquare $\pmod k$ or b is a square $\pmod q$ (these are the cases for which $c_q(b)-c_k(a)\ge 0$). Under the same assumption, the author further proves

$$\limsup_{x\to\infty} E^\pi(x;q,a,b) \ge c_q(b) - c_k(a) + \sum_{\substack{\chi \pmod q \\ \chi\ne\chi_0}} \sum_{|\gamma|<T} \frac{\overline{\chi}(b)}{\rho}e^{(\rho-1/2)u} - \sum_{\substack{\chi \pmod k \\ \chi\ne\chi_0}} \sum_{|\gamma|<T} \frac{\overline{\chi}(a)}{\rho}e^{(\rho-1/2)u}$$

for any $T>0$ and any $u\ne 0$. Finally, the author obtains an exact formula for this right-hand side with $T=\infty$ and u is negative, a version that implies that the lim sup is infinite if $a\equiv 1 \pmod k$ or $b\equiv 1 \pmod q$ but not both. The author also uses this exact formula and some shrewd explicit computation to show for the first time that $\pi(x;5,4,2)=\Omega_+(\sqrt{x}/\log x)$.

This article cites [12, 19–26].

[38] S. Knapowski and P. Turán, *Further Developments in the Comparative Prime-Number Theory VII*, Acta. Arith. **21** (1972), 193–201.

The authors show that for large enough T, there exist numbers U_1,U_2,U_3,U_4 with

$$\log\log\log T \le U_2\exp\left(-\log^{15/16} U_2\right) \le U_1 < U_2 \le T$$
$$\log\log\log T \le U_4\exp\left(-\log^{15/16} U_4\right) \le U_3 < U_4 \le T$$

such that $\theta([U_1,U_2];4,1,3) > \sqrt{U_2}$ and $\theta([U_3,U_4];4,1,3) < -\sqrt{U_4}$. In particular, there exist consecutive primes p_n and p_{n+1}, both congruent to 1 (mod 4), satisfying $\log\log\log T \le p_n < p_{n+1} \le T$.

This paper cites [20, 29, 156].

[39] ———, *On the sign changes of* $(\pi(x)-\operatorname{li} x)$*. I*, Topics in number theory (Proc. Colloq., Debrecen, 1974), North-Holland, Amsterdam, 1976, pp. 153–169. Colloq. Math. Soc. János Bolyai, Vol. 13. MR0439771

[40] ———, *On the sign changes of* $(\pi(x)-\operatorname{li} x)$*. II*, Monatsh. Math. **82** (1976), no. 2, 163–175. MR0439772

Following ideas of Littlewood, Ingham, and Skewes, the authors show unconditionally that $W(Y)\gg \log\log\log Y$ for sufficiently large Y, where the implied constants are effective. The proof itself is divided into two cases. First, supposing the existence of an RH-violating zero $\beta+i\gamma$ of $\zeta(s)$ such that $\beta\ge \frac12 + 2\log^{-1/5} Y$ and $0<\gamma\le \log^{1/5} Y$, the authors establish the much stronger lower bound

$$V_1(Y) > \frac12\left(\frac{\log Y}{2\log^{5/6} Y}\right)^{1/5} > \frac14 \log^{1/30} Y.$$

The second case, where there is no such zero, is more technical and relies as usual upon Dirichlet's box principle.

This paper cites [4, 5, 8, 10, 14, 18, 39, 147].

[41] J. Pintz, *Bemerkungen zur Arbeit: "On the sign changes of* $\pi(x)-\operatorname{li}(x)$*. II" (Monatsh. Math.* **82** *(1976), no. 2, 163–175) von S. Knapowski und P. Turán*, Monatsh. Math. **82** (1976), no. 3, 199–206 (German, with English summary). MR0439773

The author shows that there exists $c > 0$ such that $W(T) \gg (\log\log T)^c$ when T is sufficiently large. Indeed, this is a special case of a more general result that establishes many large oscillations of $\Delta(x)$: let D be sufficiently large and set $\mu = D/\log\log\log\log T$. Then there are at least $\exp\big((\log\log\log T)^{1-\mu}\big)$ sign changes of $\Delta(x)$ up to T, with oscillations as large as

$$\Delta(x) > \left(\frac{1}{2} - \frac{3\log D}{D}\right)\mu \cdot \frac{\sqrt{x}\log\log\log x}{\log x}$$

(and the negative analogue), which, when D is so large that $\mu \gg 1$, provides oscillations as large as those established by Littlewood.

This article cites [4, 10, 14, 40].

[42] C. Bays and R. H. Hudson, *The segmented sieve of Eratosthenes and primes in arithmetic progressions to* 10^{12}, Nordisk Tidskr. Informationsbehandling (BIT) **17** (1977), 121–127.

The authors describe in detail a refinement of the segmented sieve of Eratosthenes, which they call the dual sieve, designed to lower the execution time. As an illustration, they record the number of primes in the eight reduced residue classes modulo 24 (from which one can calculate the number of primes in residue classes modulo any divisor of 24) up to $10^{11}, 2\times 10^{11}, \dots, 10^{12}$. From their table, one easily observes that $\pi(x; 24, 1)$ is consistently smaller than any other $\pi(x; 24, a)$ by an amount that is very roughly $\frac{1}{2}\pi(\sqrt{x})$.

[43] J. Pintz, *On the sign changes of* $\pi(x) - \mathrm{li}(x)$, Journées Arithmétiques de Caen (Univ. Caen, Caen, 1976), Soc. Math. France, Paris, 1977, pp. 255–265. Astérisque No. 41–42. MR0447151

The author begins with a thorough summary of results on sign changes of $\Delta^\pi(x)$ and related problems; he then announces, without proof, several new results of this type. He claims that for T sufficiently large, unconditionally, $W(T) \gg \sqrt{\log T}/\log\log T$, and that there exists $c > 0$ such that every interval of the form $[T^c, T]$ contains a sign change of $\Delta^\pi(x)$; ineffectively one can narrow these intervals to the form $[Te^{-\sqrt{\log T}\log\log T}, T]$. Even if one restricts to "big sign changes", where $\Delta^\pi(x) = \Omega_\pm(\sqrt{x}\log\log\log x/\log x)$, the author asserts that the number of such sign changes up to T is $\gg \sqrt{\log T}e^{-\sqrt{\log\log T}}$ effectively and $\gg \sqrt{\log T}/(\log\log T)^2$ ineffectively; these sign changes can be localized as well, and the latter inequality even holds for large sign changes of the average of $\Delta^\pi(x)$ over intervals of length $x/\log\log x$. The author further asserts that analogous theorems can be proved for the other prime counting functions, as well as for $\pi(x; 4, 1, 3)$ and some other class of prime races.

This article cites [4, 10, 14, 19–26, 39–41].

[44] ———, *On the remainder term of the prime number formula. III. Sign changes of* $\pi(x) - \mathrm{li}(x)$, Studia Sci. Math. Hungar. **12** (1977), no. 3–4, 345–369 (1980). MR607089

This article establishes new results on the number of sign changes of $\pi(x) - \mathrm{Li}(x)$. In particular it proves the effective result

$$W(T) \gg \frac{\sqrt{\log T}}{\log\log T}$$

and corresponding results for $W^\Pi(T)$, $W^\theta(T)$, and $W^\psi(T)$ with the same lower bound. Moreover, it establishes that there is necessarily a sign change in the interval $[T, T\exp(63\sqrt{\log T}\log\log T)]$ for T large enough in each of these cases, although the lower bound on such T is effectively computable only in the $\Pi(x)$ and $\psi(x)$ versions.

Under RH, Ingham's result from 1936 gives in fact a stronger localization theorem. The author here uses Turán's method, and in particular a result of Sós–Turán, to achieve a result under the assumption that RH fails. Ingham's idea to use Fejér kernels is also applied to prove the effective lower bound on $W(T)$ in the absence of an effective localization result.

This article cites [4, 8, 10, 14, 17, 18, 39–41].

[45] C. Bays and R. H. Hudson, *Details of the first region of integers* x *with* $\pi_{3,2}(x) < \pi_{3,1}(x)$, Math. Comp. **32** (1978), no. 142, 571–576. MR0476616

The authors determine that $x = 608{,}981{,}813{,}029$ is the smallest x such that $\pi(x; 3, 2, 1) = -1$. A faster version of a previous program of theirs (which had run up to 2.5×10^{11}) was used to find this sign change. The authors provide graphs of $\pi(x; 3, 2, 1)$ near this first sign change; they highlight that $\pi(x; 3, 2, 1)$

becomes negative at two separate regions near the sign change, before taking on values shortly after that are much more positive. The authors observe that neither $\pi(x;3,2,1)$ nor $\pi(x;4,3,1)$ becomes very negative near the occurence of its first negative values; in attempts to determine a smaller Skewes number, consequently, they recommend evaluation of $\Delta^\pi(x)$ in regular intervals in order to not miss a "shallow" sign change.

This paper cites [1, 4, 15, 16, 28, 47, 147, 176, 202].

[46] J. Pintz, *On the remainder term of the prime number formula. IV. Sign changes of* $\pi(x)-\mathrm{li}(x)$, Studia Sci. Math. Hungar. **13** (1978), no. 1–2, 29–42 (1981). MR630377

This article establishes lower bounds for the number of sign changes for the error terms of classical prime-counting functions. The main theorem of this article states that for $f = \pi$, Π, θ, or ψ, there exists an absolute constant $Y(f)$ such that for $Y > Y(f)$,

$$W^f(Y) > \frac{1}{10^{11}} \frac{\log Y}{(\log\log Y)^3}.$$

Interestingly, $Y(\Pi)$ and $Y(\psi)$ are effectively computable in the author's proof, whereas $Y(\pi)$ and $Y(\theta)$ are ineffective constants.

This article cites [4, 10, 14, 17, 18, 39–41, 44].

[47] C. Bays and R. H. Hudson, *Numerical and graphical description of all axis crossing regions for the moduli* 4 *and* 8 *which occur before* 10^{12}, Internat. J. Math. Math. Sci. **2** (1979), no. 1, 111–119. MR529694

The authors of this paper determine by computation the locations where $\pi(x;4,3,1) < 0$, and where $\pi(x;8,a,1) < 0$ for any $a \in \{3,5,7\}$, for x up to 10^{12}. For $x < 10^9$, a check was made at every prime; for $10^9 \le x \le 10^{12}$, a check was made every 10^7 integers, with additional checks in between if $\pi(x;q,a,b)$ was found to be near zero. They then organize these locations into "axis-crossing regions" (ACRs) $[m,n]$, where $\pi(m;q,a,1) = \pi(n;q,a,1) = -1$ and $\pi(x;q,a,1) \ge 0$ for all x outside an ACR, with m at least twice as large as the upper bound for the previous ACR.

For $q = 4$, they find six distinct ACRs under 10^{12}. For $(q,a) = (8,5)$, they find two ACRs under 10^{12} and find no ACRs for $(q,a) = (8,3)$ or $(q,a) = (8,7)$. They compare their computations to earlier published results from Leech [15], Shanks [16], and an unpublished communication from Lehmer (dated October 29, 1975). While their results overlap with Leech and Shanks for $q = 4$ for $x \le 3\cdot 10^6$, they find that their new information contradicts a prior characterization of the ACRs as mostly consisting of sparse, tiny intervals. For example, one ACR below $x < 2\cdot 10^{10}$ contains $5\cdot 10^8$ integers where $\pi(x;4,3,1) < 0$; another ACR between $37\cdot 10^9$ and $39\cdot 10^9$ contains $1.2\cdot 10^9$ integers with $\pi(x;8,5,1) < 0$. Consequently, they argue that for large x, such regions may be more typical than sign-changes being sparse, isolated points.

This paper cites [1, 4, 15, 16, 20, 21, 42, 202].

[48] H.–J. Besenfelder, *Über eine Vermutung von Tschebyschef. I.*, J. Reine Angrew. Math. **307/308** (1979), 411–417 (German).

Using an existing explicit formula for general Mellin-transform pairs, the author shows that

$$2\sqrt{\pi y} \sum_{\substack{0<\sigma<1 \\ L(\sigma+i\gamma,\chi_{-4})=0}} e^{y(\sigma-1/2+i\gamma)^2} = \log\frac{4}{\pi} - 2\sum_{n=1}^\infty \frac{\Lambda(n)\chi_{-4}(n)}{\sqrt{n}} e^{-(\log n)^2/4y} - C_0 + 2\int_0^\infty \frac{e^{-x^2/4y+x/2}-1}{1-e^{2x}}\,dx.$$

From this identity, he proves unconditionally that

$$\lim_{x\to\infty} \sum_{p>2} (-1)^{(p-1)/2} \frac{\log p}{\sqrt{p}} e^{-(\log^2 p)/x} = -\infty.$$

(Note: the author, Hans–Joachim Besenfelder, soon changed his last name to Bentz and began to publish under that name.)

This article cites [1, 5, 6, 29].

[49] H.–J. Bentz and J. Pintz, *Quadratic Residues and the Distribution of Prime Numbers*, Monatsh. Math. **90** (1980), no. 2, 91–100.

The first section of this article offers a short, conventional history of prime number races, specifically citing Shanks's computational work and heuristics from [16] as motivation for its results. Let ℓ_1 be a quadratic residue (mod q) and ℓ_2 a quadratic non-residue (mod q). Suppose that Dirichlet L-functions (mod q) satisfying the condition that all zeroes $\beta + i\gamma$ satisfy the inequality $\beta^2 - \gamma^2 \le \frac{1}{4}$ (a "bowtie" assumption). Then for $0 \le \alpha < 1/2$,

$$\sum_{p \equiv \ell_1 \pmod q} \frac{\log p}{p^\alpha} e^{-(\log p)^2/x} - \sum_{p \equiv \ell_2 \pmod q} \frac{\log p}{p^\alpha} e^{-(\log p)^2/x} \sim \frac{c_q}{\varphi(q)} \sqrt{\pi x} \cdot e^{\frac{x}{4}(\frac{1}{2}-\alpha)^2},$$

and in particular tends to infinity. By computations of Spira, this result is unconditional for $q \le 24$.

This paper cites [1, 5–7, 15, 16, 19–26, 28–32, 34, 48, 50, 61].

[50] H.–J. Besenfelder, *Über eine Vermutung von Tschebyschef. II.*, J. Reine Angew. Math. **313** (1980), 52–58.

[TO POLISH] Besenfelder (1980), in the beginning of his paper noted that there is a small typo in Turan's (1971) paper titled *"Commemoration on Stanislaw Knapowski"* which the correct form is $\lim_{x\to\infty} \sum_{p>2} (-1)^{(p-1)/2} e^{-\log^2(p/x)}$. To prove his theorem Besenfelder provided an explicit formula which is simplified as

$$\begin{aligned} \sum_{p>2} \log p \cdot p^{-\alpha} \cdot e^{-\log^2 p/4y} \cdot \chi_1(p) &= \log 4/\pi - C + \int_0^\infty \frac{e^{-x^2/4y+\alpha x} - 1}{1 - e^{2x}} dx \\ &+ \int_0^\infty \frac{e^{-x^2/4y+(1-\alpha)x} - 1}{1 - e^{2x}} dx - \sum_{p>2, n=2} \log p \cdot p^{-2\alpha} \cdot e^{-\log^2 p/y} \cdot \chi_1(p^2) \\ - \sum_{p>2, n\ge 3} \log p \cdot p^{-n\alpha} \cdot e^{-\log^2 p^n/4y} \cdot \chi_1(p^n) &- \sum_{p,n} \log p \cdot p^{-n(1-\alpha)} \cdot e^{-\log^2 p^n/4y} \cdot \chi_1(p^n) \\ &- \sum_{\rho(x_1)} {}^{*} 2\sqrt{\pi y} \cdot e^{y(\rho-1/2)^2} \end{aligned} \tag{5.1}$$

which arises after applying $F(x) = e^{-x^2/4y+(1/2-\alpha)x}$ for $\alpha, y \in R$ and $y > 0$ in the initial explicit formula. In this explicit formula, $\chi_1(p) = (-1)^{(p-1)/2}$, $\chi_1(p^2) = +1$, and the star next to the summation shows that the roots of ρ are ordered by growing amount of their ordinates γ. Besenfelder showed that regardless of any assumptions for $0 \le \alpha \le 1/2$ we have

$$\lim_{x\to\infty} \sum_{p>2} (-1)^{(p-1)/2} \cdot \log p \cdot p{-}\alpha e^{-\log^2 p/4y} = -\infty.$$

Since the obtained equation is invariant under the substitution of $\alpha \to 1-\alpha$, the limit holds for any $0 \le \alpha \le 1$, therefore only the prove of the case when $0 \le \alpha \le 1/2$ is sufficient. There exist a typo in the second remark when defining the partial multiplication function $g(p)$. The correct form is $g(p) = \frac{e^{-p/x}}{e^{-(\log^2 p)/x}}$.
This article cites [1, 5–7, 25, 29, 48, 187, 188].

[51] R. H. Hudson, *A common combinatorial principle underlies Riemann's formula, the Chebyshev phenomenon, and other subtle effects in comparative prime number theory. I*, J. Reine Angew. Math. **313** (1980), 133–150. MR552467

In this article, the author outlines a combinatorial principle that seeks to explain various effects and biases in comparitive prime number theory. He highlights Riemann's original explicit formula $\pi(x) \sim \mathrm{li}(x) - \frac{1}{2}\mathrm{li}(x^{1/2}) - \frac{1}{3}\mathrm{li}(x^{1/3}) + \cdots$ and connects it to Chebyshev's observation, which can be seen as approximating $\pi(x; 4, 3, 1)$ by half the number of prime squares. Arguing from a generalisation of an exact formula of Meissel, the author deduces, in the example of primes (mod 4), that an "excess" in the number of integers of the form pq, where p and q are prime, in the class 1 (mod 4) must result in a corresponding "deficiency" in the number of primes of exactly this magnitude, that is, half the number

of prime squares. A combinatorial observation gives a reason for such an excess: in counting integers that are the product of two primes from a set, products of distinct primes are counted twice (as pq and qp), while the prime squares are not. The author then provides similar arguments for why cubic and higher order effects should exist. Along with describing the combinatorial principle in generality, he provides details of some numerical investigations into these effects.

This article cites [1, 4, 16, 28, 29, 37].

[52] W. R. Monach, *Numerical Investigation of Several Problems in Number Theory*, ProQuest LLC, Ann Arbor, MI, 1980. Thesis (Ph.D.), University of Michigan. MR2631002

[53] H. L. Montgomery, *The zeta function and prime numbers*, Proceedings of the Queen's Number Theory Conference, 1979 (Kingston, Ont., 1979), Queen's Papers in Pure and Appl. Math., vol. 54, Queen's Univ., Kingston, Ont., 1980, pp. 1–31. MR634679

Section 3 of this article examines random variables of the form $X = \sum_{k=1}^{\infty} r_k \sin(2\pi\theta_k)$ for $\{r_k\}$ a decreasing ℓ^2 sequence, where the θ_k are independently uniformly distributed on $\mathbb{R}/\mathbb{Z}$. The author establishes, for any integer $K \ge 1$, the bounds

$$P\biggl(X \ge 2\sum_{k=1}^{K} r_k\biggr) \le \exp\Biggl(-\frac{3}{4}\biggl(\sum_{k=1}^{K} r_k\biggr)^2\biggl(\sum_{k=K+1}^{\infty} r_k^2\biggr)^{-1}\Biggr)$$
$$P\biggl(X \ge \frac{1}{2}\sum_{k=1}^{K} r_k\biggr) \ge \frac{1}{2^{40}}\exp\Biggl(-100\biggl(\sum_{k=1}^{K} r_k\biggr)^2\biggl(\sum_{k=K+1}^{\infty} r_k^2\biggr)^{-1}\Biggr);$$

in addition, if δ is sufficiently small and $\sum_{k\colon r_k>\delta}(r_k-\delta) \ge V$, then

$$P(X \ge V) \ge \frac{1}{2}\exp\biggl(-\frac{1}{2}\sum_{k\colon r_k>\delta}\log\frac{\pi^2 r_k}{2\delta}\biggr).$$

These results can be applied to the limiting logarithmic distribution function of $E^{\psi}(x)$, which (assuming RH and LI) is the same as the distribution of the random variable $Y = \sum_{\gamma>0}\frac{2}{|\rho|}\sin(2\pi\theta_\rho)$. In particular, the second result implies that there exist constants $0 < c_1 < c_2$ such that

$$\exp\bigl(-c_2\sqrt{v}e^{\sqrt{2\pi v}}\bigr) \le P(Y > v) \le \exp\bigl(-c_1\sqrt{v}e^{\sqrt{2\pi v}}\bigr),$$

which suggests the conjecture

$$\limsup\frac{E^{\psi}(x)}{(\log\log\log x)^2} = \frac{1}{2\pi} \quad\text{and}\quad \liminf\frac{E^{\psi}(x)}{(\log\log\log x)^2} = -\frac{1}{2\pi}.$$

[54] J. Pintz, *Oscillatory properties of $M(x) = \sum_{n\le x}\mu(n)$. II*, Studia Sci. Math. Hungar. **15** (1980), no. 4, 491–496. MR688630

[55] ———, *On the remainder term of the prime number formula. I. On a problem of Littlewood*, Acta Arith. **36** (1980), no. 4, 341–365. MR585891

This article contains explicit oscillation results under the assumption that RH is false. Suppose that $\rho_0 = \beta_0 + i\gamma_0$ is a nontrivial zero of $\zeta(s)$. Let $0 < \varepsilon \le 0.02$, and set $A = 40000\varepsilon^{-2}\log\gamma_0$. Then for H sufficiently large in terms of ρ_0, there exist x_+ and x_- in the interval $[H, H^A]$ such that

$$\Delta^{\pi}(x_+) > (1-\varepsilon)\frac{x_+^{\beta_0}}{|\rho_0|\log x_+} \quad\text{and}\quad \Delta^{\pi}(x_-) < -(1-\varepsilon)\frac{x_-^{\beta_0}}{|\rho_0|\log x_-},$$

and the same for $\Delta^{\Pi}(x)$; similarly, the result holds without the factor $\log x$ in the denominator for $\Delta^{\psi}(x)$ and $\Delta^{\theta}(x)$. In all these theorems, if in addition $\beta_0 > \frac{1}{2}+\varepsilon$ and γ_0 is sufficiently large in terms of ε, then by replacing the factor $(1-\varepsilon)/|\rho_0|$ by the smaller $1/\gamma_0^{1+\varepsilon}$, the localization can be improved to the interval $[H, H^{1+\varepsilon}]$. Consequently, $W^f(T)/\log\log T$ tends to infinity for each of the four functions $f \in \{\pi, \Pi, \theta, \psi\}$ (the case where RH is true having been handled by Ingham [10]).

This article cites [4, 10, 19–26, 28–32, 39, 53].

[56] ———, *On the remainder term of the prime number formula. II. On a theorem of Ingham*, Acta Arith. **37** (1980), 209–220. MR598876

This article investigates the connection between the zero free region of $\zeta(s)$ and the size of the remainder term in the prime number theorem. Let $\eta\colon [1,\infty) \to (0,\frac12]$ be a continuous, decreasing function, and suppose that $\zeta(s)$ does not vanish when $\sigma > 1-\eta(|t|)$. If we define $\omega(x) = \min_{t\ge1}(\eta(t)\log x + \log t)$, then for any $0<\varepsilon<1$,

$$\Delta^\psi(x) \ll_\varepsilon x/e^{(1-\varepsilon)\omega(x)},$$

and the same is true for $\Delta^\theta(x)$ and $\Delta^\Pi(x)$ and $\Delta^\pi(x)$. This is an improvement of a result of Ingham [147, Theorem 22], which had a factor of $\frac12$ in the exponent (and additional conditions upon η). In particular, when combined with a 1960/61 theorem of Staś, this result provides a nearly lossless relationship between zero-free regions for $\zeta(s)$ and error terms in the prime number theorem. It follows that an Ω-theorem for any of the four error terms given above actually implies $\Omega_\pm$-theorems, of the same order of magnitude up to an ε in the exponent, for all four error terms, an implication that seems extremely difficult to prove directly.

This article cites [4, 19–26, 28–32, 39, 53, 147].

[57] ———, *On the remainder term of the prime number formula. V. Effective mean value theorems*, Studia Sci. Math. Hungar. **15** (1980), no. 1–3, 215–223. MR681441

For any of the functions $f \in \{\pi,\Pi,\theta,\psi\}$, the author establishes lower bounds for the integrated absolute error term $\Delta^f_{|1|}(x)$. The main theorem of this article states that if $\beta_0+i\gamma_0$ is a zero of the Riemann zeta function, then $\Delta^f_{|1|}(Y)/Y \ge Y^{\beta_0}e^{-2\sqrt{\log Y}(\log\log Y)^2}$ when Y is sufficiently large in terms of γ_0. The author sketches a modification of the proof that yields the stronger lower bound $\Delta^f_{|1|}(Y) \ge Y^{\beta_0}e^{-18(\log Y)^{1/3}(\log\log Y)^{4/3}}$.

This article cites [44, 46, 55, 56, 58, 144, 150, 153, 161, 162, 195].

[58] ———, *On the remainder term of the prime number formula. VI. Ineffective mean value theorems*, Studia Sci. Math. Hungar. **15** (1980), no. 1–3, 225–230. MR681442

This article concerns the absolute averages $\Delta_{|1|}$ of various standard error terms for prime counting functions. When Y is sufficiently large (ineffectively), the author proves that

$$\Delta^\pi_{|1|}(Y) > 0.62\frac{Y^{3/2}}{\log Y}, \qquad \Delta^\Pi_{|1|}(Y) > 9\cdot10^{-5}\frac{Y^{3/2}}{\log Y},$$
$$\Delta^\theta_{|1|}(Y) > 0.62Y^{3/2}, \qquad \Delta^\psi_{|1|}(Y) > 10^{-4}Y^{3/2}.$$

Thanks to work of Cramér [144], if RH is true then these bounds are best possible up to the leading constants (and even those constants are not too far off). Under RH, the author can improve some of these constants and also better localize the implied large values of the error terms; indeed, the lower bounds for the $\Delta_{|1|}$ are derived from the existence of large oscillations of the error terms, rather than the other way around.

This article cites [10, 57, 144].

[59] ———, *On the sign changes of $M(x)=\sum_{n\le x}\mu(n)$*, Analysis **1** (1981), no. 3, 191–195. MR660714

[60] H.–J. Bentz, *Discrepancies in the Distribution of Prime Numbers*, J. Number Theory **15** (1982), 252–274.

For $0\le\alpha<\frac12$, the author shows unconditionally that

$$\sum_p \chi_{-4}(p)\frac{\log p}{p^\alpha}e^{-(\log x)^2/p} \sim -\frac{\sqrt{\pi x}}{2}e^{x(1-2\alpha)^2/16};$$

when $\alpha=\frac12$, the right-hand side must be replaced by $\frac14\sqrt{\pi x}$. Both results remain valid if χ_{-4} is replaced by χ_{-3}. These results can be interpreted as comparing (in a specific way) the residue class 1 to the other reduced residue class modulo 4 or 3. Analogously, when $\alpha=\frac12$, the author establishes the same result when comparing 1 (mod 8) to another reduced residue class (mod 8); if two reduced residue classes (mod 8) are compared, the resulting expression is bounded. The author asserts that the required hypotheses on zeros of relevant Dirichlet L-functions is that they do not vanish in the "bowtie" $\{s\colon \sigma>0,\ 0<|t|<|\sigma-\frac12|\}$. The author also presents some numerical data concerning the prime number race (mod 3).

This paper cites [5–7, 15, 16, 19–26, 28–32, 34, 38, 48–50, 147, 199].

[61] H.–J. Bentz and J. Pintz, *Über das Tschebyschef-Problem*, Resultate Math. **5** (1982), no. 1, 1–5 (German). MR662791

[62] J. Pintz, *Oscillatory properties of* $M(x) = \sum_{n\le x} \mu(n)$*. I*, Acta Arith. **42** (1982/83), no. 1, 49–55. MR678996

This article is concerned with oscillations in the Mertens sum. The natural difficulty of this problem comes from the fact that the explicit formula for $M(x)$ contains terms of the form $x^\rho/\rho\zeta'(\rho)$, which are more difficult to handle than the terms x^ρ/ρ appearing in the explicit formula for $\Delta^\psi(x)$. The author proves that if $\rho_0 = \beta_0 + i\gamma_0$ is a zero of $\zeta(s)$, then for $Y > e^{|\gamma_0|+4}$,

$$\frac{1}{Y}\int_{Y/(100\log Y)}^{Y} |M(x)|\,dx > \frac{1}{6|\rho_0|^3}Y^{\beta_0} \quad\text{and}\quad \max_{x\le Y}|M(x)| \ge \frac{1}{6|\rho_0|^3}Y^{\beta_0}.$$

Consequently, using the first zeta zero $\frac12 + i\gamma_1$ with $\gamma_1 \approx 14.13$,

$$\max_{Y/(100\log Y)\le x\le Y} |M(x)| \ge \frac{1}{17000}\sqrt{Y}$$

for $Y \ge 2$; the constant $1/17000$ can be improved but not enough to disprove the Mertens conjecture.

This paper cites [35, 153].

[63] ______, *Oscillatory properties of the remainder term of the prime number formula*, Studies in pure mathematics, Birkhäuser, Basel, 1983, pp. 551–560. MR820251

The author establishes two theorems that improve and simplify prior work of Turán and of Ingham. The first theorem states that if $\zeta(\beta_0 + i\gamma_0) = 0$, then for T sufficiently large in terms of γ_0, there exists an $x \in [T^{1/4}, T]$ for which $|\Delta^\psi(x)| \gg_{\gamma_0} x^{\beta_1}$ (with an explicit dependence on γ_0). The second theorem assumes that $\zeta(s) \ne 0$ in a region of the shape $\sigma \ge 1 - \eta(t)$ where η is continuous and decreasing, and, defining $\omega(x) = \min_{t\ge0}(\eta(t)\log x + \log t)$, concludes that $\Delta^\psi(x) = \Omega(x/e^{54\omega(x)})$. The main tool is the powersum estimate of Sós and Turán. Near-optimal improvements (by the author) of these two results appeared slightly earlier [55, 56].

This article cites [2, 55, 56, 147, 153, 154, 157].

[64] J. Kaczorowski, *On sign-changes in the remainder-term of the prime-number formula. I*, Acta Arith. **44** (1984), no. 4, 365–377. MR777013

This article establishes a lower bound on the growth of the number of sign changes of $\Delta^\psi(x)$ and $\Delta^\Pi(x)$. Specifically, the author proves that $W^\psi(T) \ge \frac{\gamma_1}{4\pi}\log T$ (and the same for $W^\Pi(x)$) when T is sufficiently large (effectively), where $\gamma_1 \approx 14.1347$. The key technique used in the paper is to bound $W^\psi(T)$ below by $W\big(\Delta_n^\psi; T\big)$, the number of sign changes of repeated logarithmic integrals of $\Delta^\psi(x)$; using the fact that the second-lowest nontrivial zero of $\zeta(s)$ has imaginary part exceeding 15, the author derives an explicit formula for $W\big(\Delta_n^\psi; T\big)$ when $n \asymp \log T$ is suitably chosen.

This paper cites [4, 5, 8–10, 14, 17, 18, 39–41, 44, 46, 137, 147, 176, 196].

[65] J. Pintz and S. Salerno, *Irregularities in the distribution of primes in arithmetic progressions. I*, Arch. Math. (Basel) **42** (1984), no. 5, 439–447. MR756697

Assuming a finite Riemann-Piltz conjecture, the authors show that when Y is sufficiently large,

$$\int_{Y^{1-7/\lambda}}^{Y} \psi(x; q, \ell_1, \ell_2)\,\frac{dx}{x} \gg \sqrt{Y}\exp\left(-\frac{2\log Y}{\lambda} - c_3 q\lambda\log^2 Y\right)$$

(and the same for Π in place of ψ) for any λ satisfying

$$\frac{\sqrt{\log Y}}{\sqrt{q}\log\log Y} < \lambda < \frac{c_2\log Y}{q(\log\log Y)^2}.$$

Essentially any such choice of λ improves upon analogous results of Knapowski [162, 166, 168]. The proof also works for ψ replaced by θ or π, but only if ℓ_1 and ℓ_2 are both quadratic nonresidues (mod q).

This paper cites [31, 162, 166, 168, 214].

[66] ———, *Irregularities in the distribution of primes in arithmetic progressions. II*, Arch. Math. (Basel) **43** (1984), no. 4, 351–357. MR802311

The authors elaborate on their work in [65] to handle prime number races where a bias is present. Again assuming a finite Riemann-Piltz conjecture, they show that when Y is sufficiently large,

$$\frac{1}{Y}\int_{Y^{1-7/\lambda}}^{Y} |\pi(x;q,\ell_1,\ell_2)|\,dx \geq \sqrt{Y}\exp\left(-\frac{9\log Y}{\lambda} - c_3 q\lambda(\log\log Y)^2\right),$$

(and the same for θ in place of π) for any λ satisfying

$$\frac{\sqrt{\log Y}}{\sqrt{q}\log\log Y} < \lambda < \frac{c_2\log Y}{q\,(\log\log Y)^2}.$$

They first deal with the case when both ℓ_1 and ℓ_2 are quadratic residues (mod q), using an explicit formula involving zeros of both $L(s,\chi)$ and $L(2s,\chi)$. In the remaining case when ℓ_1 is a residue and ℓ_2 is a nonresidue, there is an additional term corresponding to the pole of $L(2s,\chi_0)$ at $s = \frac{1}{2}$.

This paper cites [65, 162, 166, 168].

[67] J. Pintz, *On the partial sums of the Möbius function*, Topics in classical number theory, Vol. I, II (Budapest, 1981), Colloq. Math. Soc. János Bolyai, vol. 34, North-Holland, Amsterdam, 1984, pp. 1229–1250. MR781183

[68] ———, *On the remainder term of the prime number formula and the zeros of Riemann's zeta-function*, Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983), Lecture Notes in Math., vol. 1068, Springer, Berlin, 1984, pp. 186–197. MR756094

This paper is primarily a summary of the results to be proved in the series [54–59] by the author. The main functions of interest are $S(x) = \max_{0\leq u\leq x}|\Delta^\psi(u)|$ and $\Delta^\psi_{|1|}(x) = \int_0^x |\Delta^\psi(u)|\,du$. The following theorem is proved: Define $\omega(x) = \log\frac{x}{Z(x)}$, where $Z(x) = \max_\rho \frac{x^\beta}{|\gamma|}$. Then

$$\log\frac{x}{S(x)} \sim \log\frac{x^2}{\Delta^\psi_{|1|}(x)} \sim \omega(x).$$

In particular, this implies that $S(x)$ and $\frac{1}{x}\Delta^\psi_{|1|}(x)$ are close in value, that is, the mean and maximum of $|\Delta^\psi(u)|$ are close. The proof uses a zero-density theorem of Carlson (for the upper bounds) and the power-sum method (for the lower bounds).

This article cites [4, 14, 17, 18, 39, 40, 54–59, 64, 147, 161, 163].

[69] ———, *Oscillatory properties of $M(x) = \sum_{n\leq x}\mu(n)$. III*, Acta Arith. **43** (1984), no. 2, 105–113. MR736725

By refining the proof method in his previous work [54], the author proves that if $\rho_0 = \beta_0 + i\gamma_0$ is a zero of $\zeta(s)$, then when Y is sufficiently large in terms of γ_0,

$$\max_{x\in[Y\exp(-5(\log\log Y)^{5/2}),Y]}\frac{M(x)}{x^{\beta_0}} > \frac{1}{48|\rho_0|^3} \quad\text{and}\quad \min_{x\in[Y\exp(-5(\log\log Y)^{5/2}),Y]}\frac{M(x)}{x^{\beta_0}} < -\frac{1}{48|\rho_0|^3}.$$

This article cites [46, 54, 62, 169, 174, 178, 179].

[70] J. Pintz and S. Salerno, *On the comparative theory of primes*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **11** (1984), no. 2, 245–260. MR764945

The authors obtain new estimates on $\psi(x;q,\ell_1,\ell_2)$ for arbitrary residues ℓ_1,ℓ_2. Assuming GRH($cq^2\log^6 q, A(q)$), the authors prove that for Y sufficiently large, there exists

$$x \in \left[Y\exp\left(-\frac{cq}{\sqrt{A(q)}}(\log Y)^{1/2}(\log\log Y)^{3/2}\right), Y\right]$$

such that

$$\psi(x;q,\ell_1,\ell_2) > \sqrt{Y}\exp\left(-\frac{cq}{\sqrt{A(q)}}(\log Y)^{1/2}(\log\log Y)^{3/2}\right).$$

This is an improvement over the work of Knapowski and Turan both in the localization and in the lower bound. The authors improve the power-sum bounds used by Knapowski and Turan to prove their results.

This article cites [19–26, 140, 173].

[71] J. Kaczorowski, *On sign-changes in the remainder-term of the prime-number formula. II*, Acta Arith. **45** (1985), no. 1, 65–74. MR791085

The author proves unconditionally that $W^\pi(T) \gg \log T$, though with an ineffective constant, and the same for $W^\theta(T)$. He also proves unconditionally that $\liminf_{T\to\infty} W^\psi(T)/\log T \ge \gamma(\Theta)/\pi$, where $\gamma(\Theta)$ is the smallest $\gamma > 0$ such that $\zeta(\Theta + i\gamma) = 0$ (or $\gamma(\Theta) = \infty$ if Θ is not attained); this improves a result of Pólya [8], which had lim sup in place of lim inf. He remarks that if RH is false, then the proof of this latter result can be extended to $W^\theta(T)$ and (with a bit more difficulty) to $W^\Pi(T)$ and $W^\pi(T)$. As in a previous paper, the proofs of both theorems make use of the iterated averages $\underline{\Delta}^f_n(x)$.

This paper cites [4, 5, 8, 10, 17, 18, 39–41, 64, 196].

[72] J. Kaczorowski and J. Pintz, *Oscillatory properties of arithmetical functions. I*, Acta Math. Hungar. **48** (1986), no. 1–2, 173–185. MR858395

In this article, the authors improve upon and extend results of Landau [3], Pólya [8], and Grosswald [173]. Given a Dirichlet intergal $F(s) = \int_{x_0}^\infty f(x)x^{-s-1}\,dx$ converging on a right half-plane $\{\sigma > \theta\}$, with a continuation to a larger half-plane except for perhaps countably many poles or logarithmic singularities (in a precise sense), the authors show that $\liminf_{T\to\infty} \frac{W(f,T)}{\log T} \ge \frac{\gamma}{\pi}$ where $\gamma = \inf\{|t| \colon F(s) \text{ is not regular at } \theta + it\}$. This result implies $\gg \log T$ sign changes of the functions $M(x)$ and $\sum_{n\le x} \mu_k(n) - x/\zeta(k)$, as well as of $\psi(x; q, \ell_1, \ell_2)$ assuming HC. For a slightly more restricted class of functions, the authors prove an effective version of a similar result, again guaranteeing $\gg \log T$ sign changes (essentially using a single singularity of $F(s)$) but now with effective constants.

This paper cites [3, 8, 23, 24, 35, 54, 64, 71, 173, 177, 196].

[73] G. Robin, *Irrégularités dans la distribution des nombres premiers dans les progressions arithmétiques*, Ann. Fac. Sci. Toulouse Math. (5) **8** (1986/87), no. 2, 159–173.

This article examines, assuming HC, the asymptotic behaviour of the weighted average $\mathcal{P}(x) = \sum_{n\le x} \mathring{\Delta}(n; k, \ell) n^{-\alpha} \log^\beta n$ where α and β are fixed real numbers. If GRH is false, then $\mathcal{P}(x) \ll 1 + x^{1-\alpha+\Theta_k} \log^{\beta-1} x$ and $\mathcal{P}(x) = \Omega_\pm(x^{1-\alpha+\Theta_k-\varepsilon})$; under the additional assumption of SA, we have $\mathcal{P}(x) = \Omega_\pm(x^{1-\alpha+\Theta_k} \log^{\beta-1} x)$ (which is thus best possible for $\alpha < 1 + \Theta_k$).

If GRH is true, the behaviour depends more significantly upon α and β. When $\alpha > \frac{3}{2}$, we have $\mathcal{P}(x) \ll 1$. When $\alpha = \frac{3}{2}$, we have $\mathcal{P}(x) \ll 1$ if $\beta < 0$, and $\mathcal{P}(x) = (1 - c_k(\ell)) \log\log x + O(1)$ if $\beta = 0$, and $\mathcal{P}(x) = (1 - c_k(\ell))(\log x)^\beta/\beta + O((\log x)^{\beta-1} \log\log x)$ if $\beta > 0$. Finally, when $\alpha < \frac{3}{2}$, we have $\mathcal{P}(x) \ll x^{3/2-\alpha} \log^{\beta-1} x$. This theorem disproves Shanks's conjecture $\sum_{n\le x} \pi(n; 4, 3, 1)n^{1/2}/\pi(n) \sim x$, as well as corresponding conjectures for other moduli. Moreover, it shows that Brent's conjecture $\sum_{n\le x} \pi(n; 4, 3, 1)/n^{1/2}\pi(n) \sim \log x$ is equivalent to GRH.

Again assuming GRH and $\alpha < \frac{3}{2}$, the author asserts that for certain moduli including $3, 4, 5, 6, 7, 8, 9, 10, 12$, there exists a constant $\alpha_{k,\ell}$ such that for $\alpha > \alpha_{k,\ell}$, when x is sufficiently large then $\mathcal{P}(x) < 0$ if ℓ is a quadratic residue and $\mathcal{P}(x) > 0$ if ℓ is a quadratic nonresidue. (It seems that this result actually holds for all moduli $k \ge 3$.) On the other hand, for some moduli including $23, 43, 67, 163$, there exists a constant $\alpha'_{k,\ell}$ such that for $\alpha < \alpha'_{k,\ell}$, we have $\mathcal{P}(x) = \Omega_\pm(x^{3/2-\alpha} \log^{\beta-1} x)$.

This article cites [5–7, 15, 16, 19–26, 28–32, 34, 37, 38, 49, 60, 193, 195, 206, 207].

[74] J. Kaczorowski and J. Pintz, *Oscillatory properties of arithmetical functions. II*, Acta Math. Hungar. **49** (1987), no. 3-4, 441–453. MR891057

The authors extend their earlier results to obtain $\gg \log T$ sign changes for functions such as $\Delta^\Pi(x; q, a)$, $\Delta^\pi(x; q, a)$ where a is a quadratic nonresidue, $\Delta^\Pi(x; q, a, b)$ where $a \not\equiv b \pmod q$, and so on. They also similarly obtain sign changes (in relatively short intervals) for the error term in the asymptotic formula for the counting function of irreducible elements in the ring of integers $\mathcal{O}_K$ of a number field K,

assuming the Dedekind zeta function of the Hilbert class field of K does not vanish on the interval $[1/2, 1)$ and has at least one simple zero in the half-plane $\sigma > 1/2$.

This article cites [10, 64, 71, 72].

[75] J. Kaczorowski, *On sign-changes in the remainder-term of the prime-number formula. III*, Acta Arith. **48** (1987), no. 4, 347–371. MR927376

This article examines $W(\Delta_e^\psi; T)$, the number of sign changes of $\Delta_e^\psi(x) = \sum_{n=1}^\infty (\Lambda(n) - 1)e^{-n/x}$ in the interval $[0, T]$. The author first proves that SA (which he calls "Ingham's condition") is equivalent to the assertion that $W\big(\Delta_e^\psi; [T, eT]\big) \ll 1$ uniformly for $T > 0$, which is further equivalent to each of $\limsup_{T\to\infty} W(\Delta_e^\psi; T) < \infty$ and $\liminf_{T\to\infty} W(\Delta_e^\psi; T) < \infty$. Assuming SA and LI($\Theta$), the author proves that $W(\Delta_e^\psi; T) \sim \kappa \log T$ as $T \to \infty$, for a constant κ (given by an explicit integral) depending on the zeros of $\zeta(s)$ on the line $\sigma = \Theta$. Finally, assuming RH, the author proves that $W(\Delta_e^\psi; T) = \frac{\gamma_1}{\pi} \log T + O(1)$, where $\gamma_1 \approx 14.13$, and indeed that these sign changes are extremely regularly spaced and correspond to oscillations that are $\gg \sqrt{x}$. These results support the author's conjecture that $W(\Delta_e^\psi; T) \sim c \log T$ as $T \to \infty$.

This article cites [5, 10, 64, 71, 165].

[76] H. J. J. te Riele, *On the sign of the difference* $\pi(x) - \mathrm{li}(x)$, Math. Comp. **48** (1987), no. 177, 323–328. MR866118

The author shows that $\pi(x) > \mathrm{li}(x)$ for some $6.62 \times 10^{370} \le x \le 6.69 \times 10^{370}$, thereby improving the previous best estimate, 1.65×10^{1165}, for Skewes's number found by Lehman [176]. Using an explicit formula for $E^\pi(e^u)$ averaged by a Gaussian kernel, Lehman had found three candidates for x near which $\pi(x) > \mathrm{li}(x)$, namely $e^{727.952}$, $e^{853.853}$, and $e^{2682.977}$. Lehman showed that $e^{2682.977}$ produced an actual example; using the zeros of $\zeta(s)$ up to height 5×10^4, found on a CYBER 205 supercomputer located at the Academic Computer Centre Amsterdam, the author shows that $e^{853.853}$ produces an actual example. The author speculates that zeros up to height 4×10^5 would be required to determine whether there is an actual example around $e^{727.952}$.

This paper cites [4, 14, 176].

[77] A. Fujii, *Some generalizations of Chebyshev's conjecture*, Proc. Japan Acad. Ser. A Math. Sci. **64** (1988), no. 7, 260–263. MR974088

[78] J. Kaczorowski, *On sign-changes in the remainder-term of the prime-number formula. IV*, Acta Arith. **50** (1988), no. 1, 15–21. MR945273

The author proves, when $\Theta > \frac{1}{2}$, that for any $\varepsilon > 0$ we have $\max_{T \le x \le (1+\varepsilon)T} |\Delta_e^\psi(x)| \gg_\varepsilon T^{\Theta-\varepsilon}$. In light of the author's previous results [75] that assumed RH, it follows that unconditionally (but ineffectively), $\max_{T \le x \le (1+\varepsilon)T} |\Delta_e^\psi(x)| \gg_\varepsilon \sqrt{T}$. The author also deduces that $W_e^\psi(T) = o(\log^2 T)$, and sketches a construction (of a "barrier") showing that this result cannot be improved without further information on the zeros of $\zeta(s)$.

This paper cites [64, 71, 75, 156, 162, 165].

[79] B. Szydło, *Über Vorzeichenwechsel einiger arithmetischer Funktionen. I*, Math. Ann. **283** (1989), 139-149 (German).

[80] ———, *Über Vorzeichenwechsel einiger arithmetischer Funktionen. II*, Math. Ann. **283** (1989), 151-163 (German).

[81] ———, *Über Vorzeichenwechsel einiger arithmetischer Funktionen. III*, Monatsh. Math. **108** (1989), 325-336 (German).

[82] J. Kaczorowski, *The k-functions in multiplicative number theory. I. On complex explicit formulae*, Acta Arith. **56** (1990), no. 3, 195–211. MR1083000

This is the first in a series of articles on the "k-functions" $k(z, \chi)$ and $K(z, \chi)$ and certain limiting values $F(x, \chi)$ of the latter (see Section 3.5 for definitions). In Section 3, the author proves that $k(z, \chi)$ can be analytically continued to a meromorphic function on the Riemann surface $\mathcal{M}$ for $\log z$, and indeed that

$$k(z,\chi) - \frac{1}{2\pi i}\frac{e^z}{e^z-1}\log z$$

is meromorphic and single-valued on $\mathbb{C}$. Indeed, the author finds all of the singularities of $k(z,\chi)$ on $\mathcal{M}$ (all simple poles) and their residues. He also establishes the functional equations

$$k(z,\chi) + e^z k(z^*,\overline{\chi}) = D(z,\chi), \qquad k(z,\chi) + \overline{k(z^c,\overline{\chi})} = e^z D(-z,\chi).$$

In Section 4, the author establishes explicit formulas for $\psi_0(x,\chi)$ and $\psi_{0r}(x,\chi)$, stated in the forms

$$(x>0) \qquad F(x,\chi) + \sum_{\substack{\beta>0 \\ L(\beta,\chi)=0}} \frac{e^{\beta x}}{\beta} = -\psi_0(e^x,\chi) - R_{\chi(-1)}(x) + B(\chi) + \begin{cases} e^x, & \text{if } \chi=\chi_0, \\ -x, & \text{if } \chi\neq\chi_0 \text{ and } \chi(-1)=1, \\ 0, & \text{if } \chi(-1)=-1. \end{cases}$$

$$(x<0) \qquad F(x,\chi) + \sum_{\substack{\beta>0 \\ L(\beta,\chi)=0}} \frac{e^{\beta x}}{\beta} = \psi_{0r}(e^{|x|},\chi) + R_{-\chi(-1)}(|x|) + C(\chi) + \begin{cases} e^x, & \text{if } \chi=\chi_0, \\ x, & \text{if } \chi\neq\chi_0 \text{ and } \chi(-1)=1, \\ 0, & \text{if } \chi(-1)=-1. \end{cases}$$

The author then shows that the left-hand side is equal to the series $\sum_\rho e^{\rho x}/\rho$ as in the classical explicit formulas for the right-hand sides.

This article cites [147].

[83] ______, *The k-functions in multiplicative number theory. II. Uniform distribution of zeta zeros*, Acta Arith. **56** (1990), no. 3, 213–224. MR1083001

Let $0 < \gamma_1 \le \gamma_2 \le \cdots$ denote the imaginary parts of non-trivial zeros of $L(s,\chi)$ in the upper half plane. In this paper Kaczorowski defines a positive Toeplitz matrix $\boldsymbol{A} = (a_{nk})$ by $a_{nk} = e^{-\gamma_k}\gamma_k^n \left(\sum_{h=1}^\infty e^{-\gamma_h}\gamma_h^n\right)^{-1}$ (for $n,k\ge 1$), and proves that for any non-zero real x, the sequence $(x\gamma_n)_{n=1}^\infty$ is $\boldsymbol{A}$-uniformly distributed (mod 1); the known result that the $x\gamma_n$ are uniformly distributed (mod 1) (in the sense of Weyl) follows as a corollary.

This article cites [82].

[84] ______, *The k-functions in multiplicative number theory. III. Uniform distribution of zeta zeros; discrepancy*, Acta Arith. **57** (1991), no. 3, 199–210. MR1105605

Continuing the previous paper in this series, the author defines an "$\boldsymbol{A}$-discrepancy"

$$D_n^*(x) = \sup_{0\le t\le 1}\left|\left(\sum_{\substack{k\ge 1 \\ \{x\gamma_k\}<t}} e^{-\gamma_k}\gamma_k^n\right)\Big/\left(\sum_{k=1}^\infty e^{-\gamma_k}\gamma_k^n\right) - t\right|.$$

He shows that $D_n^*(x) \ll (\log\log n/\log n)^{2/3}$ for every real number $x\neq 0$. Under a certain $\boldsymbol{A}$-variant of zero-density theorems for Dirichlet L-functions, he proves that $D_n^*(x) \ll 1/\log n$ and conjectures that $D_n^*(x) \sim \alpha(x)/\log n$ for some constant $\alpha(x)$.

This article cites [83].

[85] ______, *The k-functions in multiplicative number theory. IV. On a method of A. E. Ingham*, Acta Arith. **57** (1991), no. 3, 231–244. MR1105608

[86] ______, *The k-functions in multiplicative number theory. V. Changes of sign of some arithmetical error terms*, Acta Arith. **59** (1991), no. 1, 37–58. MR1133236

[87] D. R. Heath-Brown, *The distribution and moments of the error term in the Dirichlet divisor problem*, Acta Arith. **60** (1992), no. 4, 389–415. MR1159354

[88] J. Kaczorowski, *A contribution to the Shanks-Rényi race problem*, Quart. J. Math. Oxford Ser (2) **44** (1993), 451-458.

[TO POLISH] In this paper, Kaczorowski examines the Shanks–Rényi prime number race modulo q, for $q \geq 3$, with particular interest on what happens to the residue class of numbers congruent to 1 modulo q. The main theorem of this paper assumes GRH, and then states that for $q \geq 3$, there exist infinitely-many integers m such that

$$\pi(m; q, 1) > \max_{a \not\equiv 1 \pmod q} \pi(m; q, a).$$

Furthermore, the set of all m which satisfy the above inequality has positive lower density.

This statement is also true for the set of all m satisfying the following inequality

$$\pi(m; q, 1) < \min_{a \not\equiv 1 \pmod q} \pi(m; q, a).$$

From this theorem, the author arrives at the following conjecture, which he calls 'The Strong Race Hypothesis'. It states that if $\{a_1, a_2, \ldots, a_{\phi(q)}\}$ is the set of reduced residue classes modulo q, then for each permutation of this set, the set of integers m such that

$$\pi(m; q, a_1) < \pi(m; q, a_2) < \ldots < \pi(m; q, a_{\phi(q)})$$

has positive lower density.

Instead of proving the main theorem directly, Kaczorowski proves the following theorem, from which the main theorem follows. Again, assume GRH. Also, let $q \geq 3$ and let u be an arbitrary non-negative real number. Then, there exist constants $b_0 > 0, c_0 > 1$ that depend on u such that for every $T \geq 1$, we have

$$\#\Big\{T \leq m \leq c_0 T \colon \psi(m; q, 1) \geq \max_{a \not\equiv 1 \pmod q} \psi(m; q, a) + u\sqrt{m}\Big\} \geq b_0 T,$$
$$\#\Big\{T \leq m \leq c_0 T \colon \pi(m; q, 1) \geq \max_{a \not\equiv 1 \pmod q} \pi(m; q, a) + u\frac{\sqrt{m}}{\log m}\Big\} \geq b_0 T,$$
$$\#\Big\{T \leq m \leq c_0 T \colon \psi(m; q, 1) \leq \min_{a \not\equiv 1 \pmod q} \psi(m; q, a) - u\sqrt{m}\Big\} \geq b_0 T,$$
$$\#\Big\{T \leq m \leq c_0 T \colon \pi(m; q, 1) \leq \min_{a \not\equiv 1 \pmod q} \pi(m; q, a) - u\frac{\sqrt{m}}{\log m}\Big\} \geq b_0 T.$$

In order to prove this theorem, Kaczorowski uses k-functions, as well as the boundary values of Dirichlet series, which are discussed in Sections 2 and 3 of this paper respectively.

This article cites [16, 19, 82, 85].

[89] M. Rubinstein and P. Sarnak, *Chebyshev's bias*, Experiment. Math. **3** (1994), no. 3, 173–197. MR1329368

This is the paper that really placed in central roles the logarithmic limiting distributions and logarithmic densities of prime number races.

Assuming GRH: the authors show that

$$E_{q;a_1,\ldots,a_r}(x) = \frac{\log x}{\sqrt{x}}\big(\phi(q)\pi(x; q, a_1) - \pi(x), \ldots, \phi(q)\pi(x; q, a_r) - \pi(x)\big)$$

has a limiting logarithmic distribution $\mu_{q;a_1,\ldots,a_r}$ on $\mathbb{R}^r$. They give an exponential upper bound for the "tail" of $\mu_{q;a_1,\ldots,a_r}$ (that is, the mass assigned to the exterior of a large ball), as well as a doubly exponential lower bound for the portion of that tail lying in certain specific orthants. They note the analogous results for the race between $\pi(x)$ and $\mathrm{Li}(x)$, as well as for the race between $\pi(x; q, N)$ and $\pi(x; q, R)$; in these two-way races, it follows that $\underline{\delta}(\pi, \mathrm{Li})$, $\underline{\delta}(\mathrm{Li}, \pi)$, $\underline{\delta}_{q;R,N}$, and $\underline{\delta}_{q;N,R}$ are strictly positive.

Assuming GRH and LI: they give the formula for the Fourier transform of $\mu_{q;a_1,\ldots,a_r}$. From it they deduce that the densities $\delta_{q;a_1,\ldots,a_r}$ exist and are strictly positive. They characterize the races (all with $r \leq 3$) for which $\mu_{q;a_1,\ldots,a_r}$ is symmetric under all permutations of the coordinates. They show that $\delta_{q;a_1,\ldots,a_r}$ tends to $1/r!$ as q tends to infinity, and establish a central limit theorem for $E_{q;N,R}(x)/\sqrt{\log q}$. They also compute $\delta(\mathrm{Li}, \pi)$, and $\delta_{q;N,R}$ for $q \in \{3, 4, 5, 7, 11, 13\}$, to several decimal places.

This paper cites [4, 9, 11, 15, 16, 19, 45, 76, 87, 147, 151, 198].

[90] J. Kaczorowski, *On the Shanks-Rényi race problem mod 5*, J. Number Theory **50** (1995), 106-118.

[TO POLISH] The author establishes another result concerning the Shanks-Rényi race problem, and under GRH settles the question of the race for ψ between residue classes (mod 5). He conditionally proves that for any possible arrangement in this race, the set of integers such that this arrangement is attained (with square root gap between players) has positive lower density. More precisely, he shows (assuming GRH for the L-functions (mod 5)) that there exist positive constants c, b_0, and b_1 such that for any permutation (a_1, a_2, a_3, a_4) of $(1, 2, 3, 4)$,

$$\#\{T \le x \le c_0 T\colon \psi(x;5,a_1) > \psi(x;5,a_2) > \psi(x;5,a_3) > \psi(x;5,a_4), \\ \min_{1\le i<j\le 4} |\psi(x;5,a_i) - \psi(x;5,a_j)| \ge b_0\sqrt{x}\,\} \ge b_1 T.$$

The proof is another application of the author's theory of k-functions described in [82–86], and involves explicit calculations using the L-functions (mod 5) and exponential sums corresponding to each permutation.

This article cites [19, 82, 88].

[91] Jerzy Kaczorowski, *On the Shanks-Rényi race problem*, Acta Arith. **74** (1996), no. 1, 31–46. MR1367576

[TO POLISH] In [88], the author proved on GRH that in every Shanks-Rényi race (mod q), the set of x for which the residue class 1 (mod q) is in first place has positive lower density (as does the set of x for which 1 is last). In this paper the author gives a method for computing explicit permutations, in any given race, such that those permutations occur with positive lower density and feature the class 1 (mod q) in first or last place. He finds sufficient conditions for a given permutation of this kind to occur with positive lower density, and reduces these conditions to finite computable formulae. As an application, he gives permutations for each race with prime modulus ≤ 29 that satisfy these conditions and therefore provably occur with positive lower density. For example, modulo 13 he provides the permutation $(7, 8, 9, 2, 6, 12, 10, 11, 5, 3, 4)$ to which 1 can be appended in either first or last place.

This article cites [19, 82, 85, 88, 90, 218].

[92] Carter Bays and Richard H. Hudson, *Zeroes of Dirichlet L-functions and irregularities in the distribution of primes*, Math. Comp. **69** (2000), no. 230, 861–866. MR1651741

[TO POLISH] Bays and Hudson (2000) investigated seven widely spaced regions of integers with $\pi_{4,3}(x) < \pi_{4,1}(x)$ using conventional prime sieves. Let $b = 2^{\alpha_0} P_1^{\alpha_1} \cdots P_k^{\alpha_k}$ and $\gamma(b) = 2^{k+\beta-1}$ where $\beta = 1$ if $\alpha_0 = 0$ or 1, $\beta = 2$ if $\alpha_0 = 2$, and $\beta = 3$ if $\alpha_0 \ge 3$ in which $\gamma(b)$ denotes the ratio of quadratic non-residues of b to quadratic residues. Also, let $\sum_N(x,b)$ and $\sum_R(x,b)$ denote the number of primes $\le x$ in all progressions $bn + c$ with c a quadratic non-residue of b and in all progressions $bn + c'$ with c' a quadratic residue of b, respectively. To compute the entire distribution of $\pi_{4,3}(x) - \pi_{4,1}(x)$ including the sign change regions, in time linear in x, using zeroes of $L(s,\chi)$, and χ the non-principal character mod 4, they introduced the following theorem: For $b = 4$, q^α, or $2q^\alpha$, where q is an odd prime, and for $x \ge 2, T \ge 1$ under the GRH we have

$$\sum_N(x,b) - \sum_R(x,b) = \pi(\sqrt{x})/2 + \pi(\sqrt{x}) \sum_{0\le\gamma\le T} \frac{\sin\gamma\log x}{\gamma} + O_{x,T}\big(\frac{x(\log x + \log T)^2}{T\log x} + \frac{\sqrt{x}}{\log^2 x}\big),$$

where γ runs over the imaginary parts of the non-trivial zeroes of $L(s,\chi)$, and χ is the real non-principal character. Their computer program based on this theorem shows the accuracy with which the zeroes duplicate the distribution is satisfying, can discover all known axis crossing regions, and find probable regions up to 10^{1000}.

Their result is applicable to a variety of problems in comparative prime number theory including theoretical computations of logarithmic densities for Chebshev's bias for all moduli for which zeroes as well as the sign changes of $\mathrm{li}(x) - \pi(x)$ have been computed.

This article cites [15, 16, 42, 45, 47, 89, 90, 92].

[93] A. Feuerverger and G. Martin, *Biases in the Shanks–Rényi Prime Number Race*, Experiment. Math. **9** (2000), 535-570.

[94] N. Ng, *Limiting processes and Zeros of Artin L-Functions,* Ph.D. Thesis, University of British Columbia. (2000).

[95] J.–C. Puchta, *On large oscillations of the remainder of the prime number theorems*, Acta Math. Hungar. **87** (2000), no. 3, 213–227. MR1761276

[96] Carter Bays, Kevin Ford, Richard H. Hudson, and Michael Rubinstein, *Zeros of Dirichlet L-functions near the real axis and Chebyshev's bias*, J. Number Theory **87** (2001), no. 1, 54–76. MR1816036

[TO POLISH] By reviewing the GRH and GSH application in Rubenstein and Sarnak paper (1994), Bays et al. (2001), considered logarithmic densities to find an easier way to compute the Chebyshev's bias which made it possible to quickly approximate the bias for any modulus q for which zeros had been computed. Also, plots of $P_{q,N,R}(x)$ for all q with $h(-q) = 1$ is provided to outline the method of computing Chebyshev's bias. They showed that Chebyshev's bias depends strongly on the location of the first few zeros of $L(s, \chi_q)$ and the size of the first zero when q has a primitive root. By using the Chowla-Selberg formula they showed that the $L-$function has a relatively low first zero, especially for $L(s, \chi_{q=163})$, if $Q(\sqrt{-q})$ is an imaginary quadratic formula with class number 1. Bays et al. (2001) also analyzed the connection between low-lying zeros and some class number 3 and 5. In the end, they explored the sign changes of $\Delta_{q,a,b}(x)$ by comparing the results of Leech (1957), Lehmer (1969), Bays and Hudson (1979), Bays and Hudson (1978), and Bays and Hudson (1996).

This article cites [1, 4, 15, 16, 19, 45, 47, 51, 88–90, 92].

[97] K. Ford and S. Konyagin, *The prime number race and zeros of L-functions off the critical line*, Duke Math. J. **113** (2002), no. 2, 313–330. MR1909220

[TO POLISH] The authors show that in the absence of an ERH, it is possible that at least one of the six orderings of residues in a three-way Shanks–Rényi race does not occur later in the sequence. The authors consider arrangements, called barriers, of the zeroes of L-functions, such that at least one of the six orderings does not occur. Barriers are found for all possible three-way races. The authors highlight the fact that their results do not imply that the failure of an ERH implies that one of the orderings must not occur, and give a counterexample. Most constructions of barriers assume the failure of both an ERH and LI. In the final section a barrier is constructed with linearly independent zeros.

This article cites [1, 4, 16, 19–26, 28–32, 34, 45, 88–91].

[98] ______, *Chebyshev's conjecture and the prime number race*, IV International Conference "Modern Problems of Number Theory and its Applications": Current Problems, Part II (Russian) (Tula, 2001), Mosk. Gos. Univ. im. Lomonosova, Mekh.-Mat. Fak., Moscow, 2002, pp. 67–91. MR1985941

[TO POLISH] In this paper, the authors present nine problems that are central to the study of comparative prime number theory. The first eight are taken from or inspired by the problems listed by Knapowski and Turán in [19]. The ninth problem, entitled "Union–problems", examines the distribution of

$$\sum_{\substack{p\le x\\ p\in A}} 1 - \frac{|A|}{|B|}\sum_{\substack{p\le x\\ p\in B}} 1,$$

where k is a positive integer and A and B are disjoint subsets of reduced residue classes modulo k.

Throughout the rest of the paper, Ford and Konyagin provide an overview of what is already known about the first seven problems. Many results from earlier papers in this bibliography are presented.

This article cites [1, 3–7, 9, 11, 12, 14–16, 19–32, 34, 36–38, 42, 47, 51, 53, 76, 82, 85, 88–93, 96, 97, 100, 151, 176, 177, 186, 189, 198, 201, 202, 225, 229].

[99] ______, *The prime number race and zeros of L-functions off the critical line. II*, Proceedings of the Session in Analytic Number Theory and Diophantine Equations, Bonner Math. Schriften, vol. 360, Univ. Bonn, Bonn, 2003, pp. 40. MR2075622

[TO POLISH] The authors continue further their results on barriers from [97]. In this paper, they are primarily concerned with races in which the residue 1 is leading or trailing infinitely often, and also the number of orderings of residues that occur infinitely often in the race. Instead of configurations of zeros, the authors consider a different type of barrier which is a system of trigonometric sums. In the second section of the paper, results are given on trigonometric polynomials. In the later sections, barriers are examined with respect to races where the residue 1 is leading or trailing infinitely often, and the also the number of orderings that occur infinitely often.

This article cites [4, 19–26, 28–32, 34, 82, 88, 97, 98].

[100] G. Martin, *Asymmetries in the Shanks–Rényi prime number race*, Number theory for the millennium, II (Urbana, IL, 2000), A K Peters, Natick, MA, 2002, pp. 403–415. MR1956261

The author begins with the known values (assuming GRH and LI), for $q = 8$ and $q = 12$, of $\delta_{q;a,1}$ where $a \not\equiv 1 \pmod q$ and of $\delta_{q;a,b,c}$ where $\{a, b, c\}$ is a permutation of the three non-identity elements of $(\mathbb{Z}/q\mathbb{Z})^\times$. He investigates how one could have predicted the relative sizes of these densities using the values and conductors of the nonprincipal Dirichlet characters (mod q), by arguing by analogy with independent random variables with variances of different sizes.

The author also comments on the equality of a family of variant definitions of the logarithmic density of a set of positive real numbers, as well as some conjectures on the rarity of ties $\pi(x; q, a) = \pi(x; q, b)$.

This article cites [15, 45, 47, 51, 89, 93].

[101] N. Ng, *The distribution of the summatory function of the Möbius function*, Proc. London Math. Soc. (3) **89** (2004), no. 2, 361–389. MR2078705

[TO POLISH] In this paper, Ng presents and proves three theorems about Mertens sum $M(x)$ assuming RH and that

$$J_{-1}(T) = \sum_{0<\gamma<T} \frac{1}{|\zeta'(\rho)|^2} \ll T.$$

Under these assumptions, the first theorem states that $M(x) \ll x^{1/2}(\log x)^{3/2}$ and further that, everywhere except on a set of finite logarithmic measure, $M(x) \ll x^{1/2}(\log\log x)^{3/2}$. This theorem also asserts that

$$\int_2^Y \frac{M(x)^2}{x} dx \ll Y,$$

and that the weak Mertens conjecture,

$$\int_2^Y \left(\frac{M(x)}{x}\right)^2 dx \ll \log Y,$$

holds. The second major theorem of this paper again assumes RH and that $J_{-1}(T) \ll T$ and then states that $e^{y/2}M(e^y)$ has a limiting distribution ν on $\mathbb{R}$. In other words, for all bounded Lipschitz continuous real functions f,

$$\lim_{Y\to\infty} \frac{1}{Y} \int_0^Y f(e^{-y/2}M(e^y))dy = \int_{-\infty}^{\infty} f(x)d\nu(x).$$

Under the same two assumptions as the first two theorems, Theorem 3 states that

$$\int_0^Y \left(\frac{M(e^y)}{e^{y/2}}\right)^2 dy \sim Y \sum_{\gamma>0} \frac{2}{|\rho\zeta'(\rho)|^2}.$$

This article cites [12, 53, 87, 89, 94, 144, 147, 159, 186, 213].

[102] J.–C. Schlage–Puchta, *Sign changes of* $\pi(x; q, 1) - \pi(x; q, a)$, Acta Math. Hungar. **102** (2004), 305–320.

[TO POLISH] Using techniques taken from the author's previous work [95], this paper proves two theorems, each giving upper bounds on the first sign change and lower bounds on the number of sign changes of a single function, given some assumptions. The first assumes GRH and concerns $\pi(x; q, 1, a)$ for any $a \not\equiv 1 \pmod q$. The second assumes RH and concerns $\Delta_\pi(x)$. $W^\pi(T)$. Both theorems give upper bounds for the first sign change and lower bounds for the number of sign changes. For the first theorem, assume GRH for some modulus q, and let $q^+ = \max\{q, \exp(1260)\}$ and $f(q) = \#\{a \in \mathbb{Z}/q\mathbb{Z} \colon a^2 = 1\}$. Then there exists an $x < \exp\exp\big((q^+)^{170} + \exp(18f(q))\big)$ such that $\pi(x; q, 1, a) > 0$ for all $a \not\equiv 1 \pmod q$. Furthermore, for $F(x) = \min_{a\not\equiv 1 \pmod q} \pi(x; q, 1, a)$, $W(F; T) > \log T / \big((q^+)^{170} + \exp(18f(q))\big) - 1$. For the second theorem, assume RH. Then there exists some $x < \exp\exp\exp(16.7)$ such that $\pi(x) < \mathrm{li}(x)$. Furthermore, $W^\pi(T) > \log T / \exp\exp(16.7) - 1$.

This paper cites [9, 10, 44, 46, 76, 88, 90, 95, 176].

[103] A. Granville and G. Martin, *Prime Number Races*, Amer. Math. Monthly **113** (2006), 1–33.

[104] P. Sarnak, *Letter to Barry Mazur on 'Chebyshev's bias' for* $\tau(p)$, http://web.math.princeton.edu/sarnak/MazurLtrMay08.PDF.

Let $\lambda(p) = \tau(p)/p^{11/2}$ denote the normalized pth Fourier coefficient of the Ramanujan function $\Delta(z)$, a holomorphic cusp form of weight 12. Under various assumptions (functional equation, RH, LI) for the symmetric power L-functions associated to $\Delta(z)$, the author proves the existence of a limiting logarithmic distribution for the cumulative sum $x^{-1/2}\log x \cdot \sum_{p\le x}\lambda(p)$. This distribution has mean 1 but infinite variance, so that even though there is a bias towards being positive, the set of such x has logarithmic density $\frac12$. The author provides a similar analysis for $x^{-1/2}\log x\cdot\sum_{p\le x} a_E(p)/\sqrt{p}$, where $a_E(p)$ are the coefficients of the weight-2 cusp form attached to an elliptic curve $E/\mathbb{Q}$; here the mean is $1-2r(E)$ where $r(E)$ is the rank of E. Here the variance is (conjecturally) finite, and so the logarithmic density of the set of x for which $\sum_{p\le x} a_E(p)/\sqrt{p} > 0$ is strictly between $\frac12$ and 1 when $r(E) = 0$, but strictly between 0 and $\frac12$ when $r(E) \ge 1$. The author makes analogous remarks about the symmetric powers of these elliptic curve L-functions, where the finiteness of the variance corresponds to whether E has complex multiplication.

This article cites [89].

[105] H. G. Diamond and J. Pintz, *Oscillation of Mertens' product formula*, J. Théor. Nombres Bordeaux **21** (2009), no. 3, 523–533. MR2605532

[106] J. P. Sneed, *Prime and quasi-prime number races*, ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–University of Illinois at Urbana-Champaign. MR2753165

[107] K. Ford and J. Sneed, *Chebyshev's Bias for Products of Two Primes*, Experiment. Math. **19** (2010), 385–398.

The authors examine Chebyshev's bias for integers which are the product of two primes. Let $\pi_2(x;q,a)$ denote the number of integers n up to x such that $n \equiv a \pmod q$ and $\Omega(n) = 2$. Assume $\mathrm{ERH}_q(0)$ and also that the zeros of $L(s,\chi)$ are simple for each nonprincipal character $\chi \pmod q$. If $f(x_1,\dots,x_r)$ is the logarithmic density function of $\big(E(x;q,a_1,b_1),\dots,E(x;q,a_r,b_r)\big)$, the authors show that the logarithmic density function of $\big(E_2(x;q,a_1,b_1),\dots,E_2(x;q,a_r,b_r)\big)$ is

$$f\left(\frac{c_q(b_1)-c_q(a_1)}{2} - x_1, \dots, \frac{c_q(b_r)-c_q(a_r)}{2} - x_r\right).$$

Consequently, assuming both ERH_q and LI_q, the authors show that $\delta_2\big(\pi_2(x;q,a),\pi_2(x;q,b)\big)$ exists, and equals $\frac12$ if a and b are both quadratic residues or both quadratic nonresidues (mod q). Otherwise, if a is a quadratic nonresidue and b is a quadratic residue, then

$$1 - \delta_{q;a_1,a_2} < \delta_2\big(\pi_2(x;q,a),\pi_2(x;q,b)\big) < \frac12.$$

This article cites [1, 4, 15, 19, 89, 98, 103, 110, 131].

[108] Y. Lamzouri, *The Shanks-Rényi prime number race with many contestants*, Math. Res. Lett. **19** (2012), no. 3, 649–666. MR2998146

[109] ———, *Large deviations of the limiting distribution in the Shanks-Rényi prime number race*, Math. Proc. Cambridge Philos. Soc. **153** (2012), no. 1, 147–166. MR2943671

[110] D. Fiorilli and G. Martin, *Inequities in the Shanks-Rényi prime number race: an asymptotic formula for the densities*, J. Reine Angew. Math. **676** (2013), 121–212. MR3028758

[111] K. Ford, Y. Lamzouri, and S. Konyagin, *The prime number race and zeros of Dirichlet L-functions off the critical line: Part III*, Q. J. Math. **64** (2013), no. 4, 1091–1098. MR3151605

The authors construct "barriers" for two-way prime number races, including $\pi(x) - \mathrm{li}(x)$. They show that a certain hypothetical configuration of zeros of $L(s, \chi)$ would imply that $\{x \geq 2 \colon \pi(x; q, b) > \pi(x; q, a)\}$ has asymptotic density 0. Furthermore, they show that a certain hypothetical configuration of the zeros of $\zeta(s)$ would imply that $\{x \geq 2 \colon \pi(x) > \mathrm{li}(x)\}$ has asymptotic density 0, while a related configuration would result in the same set having asymptotic density 1.

This article cites [4, 19–26, 28–32, 34, 38, 88, 89, 97–99, 103, 106, 218].

[112] Y. Lamzouri, *Prime number races with three or more competitors*, Math. Ann. **356** (2013), no. 3, 1117–1162. MR3063909

[113] A. Akbary, N. Ng, and M. Shahabi, *Limiting distributions of the classical error terms of prime number theory*, Q. J. Math. **65** (2014), no. 3, 743–780. MR3261965

[114] D. Fiorilli, *Highly biased prime number races*, Algebra Number Theory **8** (2014), no. 7, 1733–1767. MR3272280

It is known, assuming GRH and LI, that $\delta(p; N, R)$ tends to $\frac{1}{2}$ as the prime modulus p tends to infinity. In contrast, the author shows that the analogous density $\delta(q; N, R)$ takes a set of values that is dense in $(\frac{1}{2}, 1)$, under the same two assumptions (although LI can be weakened to a mild bound on the multiplicities of zeros of $L(s, \chi)$), with large biases corresponding to highly composite moduli in a quantitative sense (including a conjecture for the asymptotic size of the analogues of Skewes's number for these races). The author proves similar results for fairly general linear combinations of reduced residue classes.

This paper cites [5–7, 9, 52, 53, 89, 93–96, 101, 103, 109, 110, 113, 130, 220, 223].

[115] ———, *Elliptic curves of unbounded rank and Chebyshev's bias*, Int. Math. Res. Not. IMRN **18** (2014), 4997–5024. MR3264673

[116] J. Büthe, *On the first sign change in Mertens' theorem*, Acta Arith. **171** (2015), no. 2, 183–195. MR3414306

Rosser and Schoenfeld observed that $\sum_{p \leq x} 1/p \geq \log\log x + M$ for $1 \leq x \leq 10^8$ (where M is the constant in Mertens's formula). The author proves that there exists an $x_0 \approx 1.91 \times 10^{215}$ such that $\sum_{p \leq x} 1/p < \log\log x + M$ for $x \in [x_0 - 6 \times 10^{103}, x_0]$. The proof is an adaptation of a method of Lehman [176] that improved upon Skewes's number, though with a different choice of kernel function.

This paper cites [9, 167, 176, 210, 260].

[117] H. Kisilevsky and M. O. Rubinstein, *Chebotarev sets*, Acta Arith. **171** (2015), no. 2, 97–124. MR3414302

[118] B. Cha, D. Fiorilli, and F. Jouve, *Prime number races for elliptic curves over function fields*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 5, 1239–1277. MR3581815

[119] D. Dummit, A. Granville, and H. Kisilevsky, *Big biases amongst products of two primes*, Mathematika **62** (2016), no. 2, 502–507. MR3521338

This article establishes permanent, relatively large biases in races involving products of two primes. The authors show that if χ is a quadratic character (mod d), then

$$\frac{\#\{pq \leq x \colon \chi(p) = \chi(q) = -1\}}{\#\{pq \leq x \colon (pq, d) = 1\}} = 1 - \left(\sum_p \frac{\chi(p)}{p} + o(1)\right)\frac{1}{\log\log x},$$

and the same statement with both minus signs changed to plus signs. In particular, the race between integers pq with $\chi(p) = \chi(q) = -1$ and those with $\chi(p) = \chi(q) = 1$ has a bias in favor of the sign of (and proportional to) $\sum_p \chi(p)/p$. For example, $\sum_p \chi_5(p)/p \approx -1.008$, and correspondingly integers pq with both p and q quadratic nonresidues (mod 5) are 41.6% more numerous up to 10^7 than random chance would suggest. The authors conjecture that there exists $d \leq x$ such that the right-hand side is as large as $1 + \log\log\log x / \log\log x$. They also note that the same proof gives a bias for the ratio

$$\sum_{\substack{p \le x \\ \chi(p)=1}} \frac{1}{p} \Big/ \sum_{\substack{p \le x \\ \chi(p)=-1}} \frac{1}{p} = 1 + \left(2 \sum_{p} \frac{\chi(p)}{p} + o(1)\right) \frac{1}{\log\log x},$$

giving a permanent bias involving only primes (in contrast to the unweighted race between $\pi(x; d, R)$ and $\pi(x; d, N)$); the proof also generalizes to products of k primes with prescribed quadratic character values (for fixed k). They remark on the possibility of counting $pq \le x$ where p and q are restricted to prescribed but arbitrary residue classes.

This article cites [107].

[120] Y. Lamzouri, *A bias in Mertens' product formula*, Int. J. Number Theory **12** (2016), no. 1, 97–109. MR3455269

[121] R. J. Lemke Oliver and K. Soundararajan, *Unexpected biases in the distribution of consecutive primes*, Proc. Natl. Acad. Sci. USA **113** (2016), no. 31, E4446–E4454. MR3624386

[122] D. J. Platt and T. S. Trudgian, *On the first sign change of $\theta(x) - x$*, Math. Comp. **85** (2016), no. 299, 1539–1547. MR3454375

The authors compute that $\Delta_\theta(x) < 0$ for $0 \le x \le 1.39 \times 10^{17}$. By partial summation, this implies that $\Delta_\pi(x) < 0$ for $2 < x \le 1.39 \times 10^{17}$. The authors also prove that there exists $x \approx 1.3971623 \times 10^{316}$ for which $\Delta_\theta(x) > 0$.

This paper cites [4, 14, 147, 176, 223, 238, 241, 260].

[123] Xianchang Meng, *The distribution of k-free numbers and the derivative of the Riemann zeta-function*, Math. Proc. Cambridge Philos. Soc. **162** (2017), no. 2, 293–317. MR3604916

[124] A. J. Harper and Y. Lamzouri, *Orderings of weakly correlated random variables, and prime number races with many contestants*, Probab. Theory Related Fields **170** (2018), no. 3-4, 961–1010. MR3773805

[125] X. Meng, *Large bias for integers with prime factors in arithmetic progressions*, Mathematika **64** (2018), no. 1, 237–252. MR3778223

[126] ______, *Chebyshev's bias for products of k primes*, Algebra Number Theory **12** (2018), no. 2, 305–341. MR3803705

[127] L. Devin, *Chebyshev's bias for analytic L-functions*, http://front.math.ucdavis.edu/1706.06394.

[128] J. Lay, *Sign changes in Mertens' first and second theorems*, http://front.math.ucdavis.edu/1505.03589.

[129] G. Martin and N. Ng, *Inclusive prime number races*, http://front.math.ucdavis.edu/1710.00088.

[130] C. Myerscough, *Application of an accurate remainder term in the calculation of residue class distributions*, http://front.math.ucdavis.edu/1301.1434.

[131] J. P. Sneed, *Lead changes in the prime number race*, Math. Comp. (to appear?).

[132] A. Shchebetov, *Chebyshev's bias computational project*, http://math101.guru/en/downloads-2/repository/.

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[133] A. Piltz, *Über die Häufigkeit der Primzahlen in arithmetischen Progressionen und über verwandte Gesetze*, Habilitationsschrift, Friedrich–Schiller–Universität Jena (1884).

Several authors cite this habilitation thesis as the first appearence of the generalized Riemann hypothesis for Dirichlet L-functions.

[134] F. Mertens, *Über eine zahlentheoretische Funktion*, SBer. Kais. Akad. Wissensch. Wien **106** (1897), 761–830 (Abt. 2a).

Reportedly, the author conjectures the bound $\left|\sum_{n\le x}\mu(n)\right| \le \sqrt{x}$ for the summatory function that now bears his name.

[135] R. D. von Sterneck, *Empirische Untersuchung über den Verlauf der zahlen-theoretischen Funktion* $\sigma(n) = \sum_{x=1}^{x=n}\mu(x)$ *im Intervalle von* 0 *bis* 150000, SBer. Kais. Akad. Wissensch. Wien **106** (1897), 835–1024 (Abt. 2a).

[136] ———, *Empirische Untersuchung ... von* 150000 *bis* 500000, SBer. Kais. Akad. Wissensch. Wien **110** (1901), 1053–1102 (Abt. 2a).

Reportedly, the author conjectures the bound $\left|\sum_{n\le x}\mu(n)\right| \le \frac{1}{2}\sqrt{x}$ for the Mertens function.

[137] E. Schmidt, *Über die Anzahl der Primzahlen unter gegebener Grenze*, Math. Annalen **57** (1903), no. 2, 195–204.

Reportedly, the author shows that $W^{\Pi}(T)$ tends to infinity with T.

[138] T. J. Stieltjes, *Correspondance d'Hermite et de Stieltjes*, Gauthier–Villars, Imprimeur–Libraire, Paris, 1905 (French).

In Lettre 79 (starting on page 160), the author states the conjectural bound $\left|\sum_{n\le x}\mu(n)\right| \le \sqrt{x}$ for the Mertens function.

[139] Edmund Landau, *Über einen Satz von Tschebyschef*, Math. Ann. **61** (1906), no. 4, 527–550 (German). MR1511360

[140] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen. 2 Bände*, Leipzig und Berlin, B. G. Teubner, 1909 (German).

In sections 201–203 (starting on page 711), the author proves that both $\pi(x) - \mathrm{Li}(x)$ and $\Pi(x) - \mathrm{Li}(x)$ are $\Omega_\pm(x^{\Theta-\varepsilon})$, where Θ is the supremum of the real parts of the zeros of $\zeta(s)$.

[141] R. D. von Sterneck, *Neue empirische Daten fiber die zahlentheoretische Funktion* $\sigma(n)$, Proc. 5th International Congress of Mathematicians, vol. 1 (1912), 341–343.

The function referred to in the title is the Mertens function $\sum_{n\le x}\mu(n)$.

[142] G. H. Hardy and J. E. Littlewood, *On an assertion of Tchebychef*, Proc. London Math. Soc. (2) **14** (1915), xv–xvi.

[143] G. Pólya, *Verschiedene Bemerkungen zur Zahlentheorie*, Jahresbericht der deutschen Math.–Vereinigung **28** (1919), 31–40.

Reportedly, the author conjectures that $\sum_{n\le x}\lambda(n) \le 0$, where $\lambda(n) = (-1)^{\Omega(n)}$ is the Liouville function.

[144] Harald Cramér, *Ein Mittelwertsatz in der Primzahltheorie*, Math. Z. **12** (1922), no. 1, 147–153, DOI 10.1007/BF01482072 (German). MR1544509

Reportedly, the author shows that $\Delta_1^\pi(x) \ll x^{3/2}/\log x$ (and the same for $\Delta_1^\Pi(x)$) and that $\Delta_1^\psi(x) \ll x^{3/2}$ (and the same for $\Delta_1^\theta(x)$).

[145] J. E. Littlewood, *Mathematical Notes: 3; on a Theorem Concerning the Distribution of Prime Numbers*, J. London Math. Soc. **2** (1927), no. 1, 41–45. MR1574052

[146] G. Pólya, *On polar singularities of power series and of Dirichlet series*, Proc. London Math. Soc. (2) **33** (1931), no. 2, 85–101. MR1576856

[147] A. E. Ingham, *The distribution of prime numbers*, Cambridge Tracts in Mathematics and Mathematical Physics. 30. London: Cambridge University Press, 1932.

[MULTIPLE EDITIONS EXIST AND ARE CITED BY VARIOUS PAPERS]

[148] B. Jessen and A. Wintner, *Distribution functions and the Riemann zeta function*, Trans. Amer. Math. Soc. **38** (1935), no. 1, 48–88. MR1501802

[149] A. Wintner, *On the Asymptotic Distribution of the Remainder Term of the Prime-Number Theorem*, Amer. J. Math. **57** (1935), no. 3, 534–538. MR1507933

[150] J. E. Littlewood, *Mathematical Notes (12): An Inequality for a Sum of Cosines*, J. London Math. Soc. **12** (1937), no. 3, 217–221, DOI 10.1112/jlms/s1-12.2.217. MR1575079

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$$\psi_k(x) = \frac{1}{k!}\sum_{n\le x}\Lambda(n)\big(\log \tfrac{x}{n}\big)^k = x - \sum_{\rho}\frac{x^\rho}{\rho^{k+1}} + \sum_{j=0}^{k}\frac{a_{k-j}}{j!}(\log x)^j$$

is the k-fold logarithmic average of $\psi(x)$, then $W(\psi_k; T) = \frac{\gamma_1}{2\pi}\log T + O_k(1)$ for $k \ge 5$, where $\gamma_1 \approx 14.135$ is the smallest ordinate of a nontrivial zero of $\zeta(s)$. The phenomenon is that the repeated averaging washes out all of the small-scale sign changes expected for $\psi(x)$ itself.

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