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ON TWO CONJECTURES IN THE THEORY OF NUMBERS.*

By A. E. Ingham.

1. Let

$$M(x) = \sum_{n \leq x} \mu(n), \qquad L(x) = \sum_{n \leq x} \lambda(n),$$

where $\mu(n)$ and $\lambda(n)$ are the arithmetical functions of Möbius and Liouville defined by

$$(1) \qquad 1/\zeta(s) = \prod_p (1 - (1/p^s)) = \sum_{n=1}^{\infty} \mu(n)/n^s$$

$$(2) \qquad \zeta(2s)/\zeta(s) = \prod_p [1 - (1/p^s) + (1/p^{2s}) - \cdots] = \sum_{n=1}^{\infty} \lambda(n)/n^s$$

$$(s = \sigma + it, \sigma > 1).$$

It was suggested by Stieltjes [**6**] and by Mertens [**1**] that $|M(x)| < x^{1/2}$ $(x > 1)$. The calculations of von Sterneck [**3, 4, 5**] confirm this up to $x = 500{,}000$, and at intervals up to $5{,}000{,}000$, and indeed with a factor $\frac{1}{2}$ on the right hand side except near $x = 200$.

It was conjectured by Pólya [**2**] that $L(x) \leq 0$ $(x \geq 2)$. This is supported by a smaller body of numerical evidence (Pólya verified it up to $x = 1{,}500$),[1] but the conjecture is attractive on account of its connection with the theory of binary quadratic forms $ax^2 + bxy + cy^2$ (a, b, c integers). Pólya showed that a positive integer m is a solution of the equation $L(m) = 0$ whenever $4m + 3$ is a prime $p > 7$ for which $h(-p) = 1$, where $h(D)$ is the number of classes of (positive definite) forms of (negative) discriminant $D(= b^2 - 4ac)$. If these were the only solutions of $L(m) = 0$, it would be easy to deduce, in view of the fact that, since $\lambda(n) = \pm 1$, $L(x)$ cannot change sign without vanishing, and of Heilbronn's theorem that $h(D) = 1$ for only a finite number of negative D, that $L(x) < 0$ at any rate for all sufficiently large x. But other solutions of $L(m) = 0$ occur within the limits of Pólya's calculations, and the argument decides nothing.

It is well known (and the proof is reproduced for completeness at the beginning of § 3) that the truth of either of the above conjectures, or more generally the truth of any one of the four inequalities

$$M(x) < Kx^{1/2}, \quad M(x) > -Kx^{1/2}, \quad L(x) < Kx^{1/2}, \quad L(x) > -Kx^{1/2},$$

* Received February 12, 1941.

[1] The verification has recently been extended to $x = 20{,}000$. See H. Gupta, "On a table of values of $L(n)$," *Proc. Indian Acad. Sci.*, Sect. A. vol 12 (1940), pp. 407-409.

for all sufficiently large x, where K is a constant, would imply that $\zeta(s)$ has all its complex zeros on the line $\sigma = \frac{1}{2}$ (i. e., that the Riemann hypothesis is true), and that all the zeros are simple. The purpose of this note is to point out a further consequence, namely that the imaginary parts of the zeros above the real axis must be linearly dependent (with rational integral multipliers). More precisely, we shall prove

THEOREM A. *If the imaginary parts $\gamma_1, \gamma_2, \cdots$ of the (distinct) zeros of $\zeta(s)$ above the real axis are connected by no relation of the type*

$$\sum_{n=1}^{N} c_n \gamma_n = 0 \quad (c_n \text{ integers not all } 0), \tag{3}$$

or by only a finite number of such relations, then, when $x \to \infty$,

$$\overline{\lim}\, x^{-\frac{1}{2}} M(x) = +\infty, \qquad \underline{\lim}\, x^{-\frac{1}{2}} M(x) = -\infty,$$
$$\overline{\lim}\, x^{-\frac{1}{2}} L(x) = +\infty, \qquad \underline{\lim}\, x^{-\frac{1}{2}} L(x) = -\infty.$$

The method of proof is similar to that by which Littlewood disproved the conjecture $\pi(x) < \operatorname{li} x$ in the theory of primes, except that the use of Phragmén-Lindelöf theorems and of 'explicit formulae' is avoided by an application of the technique introduced by Wiener in his fundamental researches on Tauberian theorems, and that Dirichlet's theorem on Diophantine approximation is replaced by Kronecker's theorem. It is this that calls for the hypothesis of linear independence of the γ_n. It would be easy to relax this hypothesis a little, but there seems no obvious way of replacing it by anything essentially easier to verify.

2. The proof of Theorem A is based on a theorem concerning Laplace integrals (Theorem 1) and on a special property of the generating functions (1) and (2) (Theorem 2).

THEOREM 1. *Let*

$$F(s) = \int_0^\infty A(u) e^{-su} du, \tag{4}$$

where $A(u)$ is absolutely integrable over every finite interval $0 \leq u \leq U$, and the integral is convergent in some half-plane $\sigma > \sigma_1 \geq 0$.

Let $A^*(u)$ be a real trigonometrical polynomial

$$A^*(u) = \sum_{-N}^{N} \alpha_n e^{i\gamma_n u} \qquad (\gamma_n \text{ real}, \gamma_{-n} = -\gamma_n, \alpha_{-n} = \bar{\alpha}_n),$$

and let

$$F^*(s) = \int_0^\infty A^*(u) e^{-su} du = \sum_{-N}^{N} \frac{\alpha_n}{s - i\gamma_n} \qquad (\sigma > 0).$$

Suppose that $F(s) - F^*(s)$ (suitably defined outside the half-plane $\sigma > \sigma_1$) is regular in the region $\sigma \geq 0$, $-T \leq t \leq T$, for some $T > 0$ (or, more generally, continuous in this region and regular in the interior).

Then, when $u \to \infty$ (T fixed)

$$(5) \qquad \underline{\lim}\, A(u) \leq \underline{\lim}\, A^*_T(u) \leq \overline{\lim}\, A(u),$$

where

$$A^*_T(u) = \sum_{|\gamma_n| < T} [1 - (|\gamma_n|/T)] \alpha_n e^{i\gamma_n u} = \alpha_0 + 2\Re \sum_{0 < \gamma_n < T} [1 - (\gamma_n/T)] \alpha_n e^{i\gamma_n u}.$$

It is enough to prove the inequality for the lower limits, since the other may be deduced by changing signs. We may suppose that $\underline{\lim}\, A(u) > -\infty$ (since otherwise there is nothing to prove), and indeed that $\underline{\lim}\, A(u) > 0$ [by adding a constant to $A(u)$ and to $A^*(u)$]. Then $A(u) > 0$ for $u > u_0$ (say), and it follows from a well known theorem of Landau that the integral in (4) is convergent, and the relation (4) true, for $\sigma > 0$, since $F(s)$ is regular along the positive real axis $s > 0$, this being true of $F^*(s)$ (obviously) and of $D(s) = F(s) - F^*(s)$ (by hypothesis). Thus we have (on our suppositions)

$$(6) \qquad D(s) = \int_0^\infty A(u) e^{-su} du - \int_0^\infty A^*(u) e^{-su} du \qquad (\sigma > 0).$$

Let $\quad k(t) = k_T(t) = 1 - (|t|/T) \quad (|t| < T); \quad = 0 \quad (|t| \geq T),$

$$K(v) = K_T(v) = T[(\sin Tv/2)/(Tv/2)]^2,$$

so that

$$\int_{-T}^{T} k(t) e^{-itv} dt = K(v), \qquad \int_{-\infty}^{\infty} K(v) e^{itv} dv = 2\pi k(t).$$

Multiplying (6) by $k(t)e^{it\omega}$ $(\omega > 0)$ and integrating over $-T \leq t \leq T$, we obtain [since the integrals in (6) are uniformly convergent over this range for fixed $\sigma > 0$]

$$\int_{-T}^{T} D(\sigma + it) k(t) e^{it\omega} dt = \int_0^\infty A(u) K(u - \omega) e^{-\sigma u} du$$
$$- \int_0^\infty A^*(u) K(u - \omega) e^{-\sigma u} du \qquad (\sigma > 0).$$

The first and third integrals here are continuous functions of σ for $\sigma \geq 0$, the first because $D(s)$ is continuous for $\sigma \geq 0$, $-T \leq t \leq T$, and the third because the integral is (absolutely) convergent when $\sigma = 0$ [the integrand then being $O(u^{-2})$ as $u \to \infty$ (T and ω fixed)]. The second integral, having a non-negative integrand for $u > u_0$, is continuous for $\sigma \geq 0$ or tends to $+\infty$

when $\sigma \to +0$ (according as it is convergent or divergent to $+\infty$ when $\sigma = 0$). The latter alternative is, however, excluded by the behaviour of the other two integrals, and we obtain, on making $\sigma \to +0$,

$$\int_{-T}^{T} D(it)k(t)e^{it\omega}dt = \int_0^{\infty} A(u)K(u-\omega)\,du - \int_0^{\infty} A^*(u)K(u-\omega)\,du.$$

Now let $\omega \to \infty$. The first integral tends to 0 by the Riemann-Lebesgue theorem, and we obtain

$$(7) \qquad \int_0^{\infty} A(u)K(u-\omega)\,du = \int_0^{\infty} A^*(u)K(u-\omega)\,du + o(1)$$

as $\omega \to \infty$ (T fixed).

Since $K(v) \geq 0$ the first integral here is

$$\geq - \max_{-\omega \leq v \leq \xi - \omega} K(v) \cdot \int_0^{\xi} |A(u)|\,du + \underset{u \geq \xi}{\underline{\mathrm{bd}}}\, A(u) \cdot \int_{\xi-\omega}^{\infty} K(v)\,dv \quad (0 < \xi < \omega),$$

whence we deduce, making $\omega \to \infty$ and then $\xi \to \infty$,

$$(8) \qquad \varliminf_{\omega\to\infty} \int_0^{\infty} A(u)K(u-\omega)\,du \geq \varliminf_{u\to\infty} A(u) \cdot \int_{-\infty}^{\infty} K(v)\,dv = \varliminf_{u\to\infty} A(u) \cdot 2\pi k(0).$$

Also, the second integral in (7) is

$$(9) \qquad \begin{aligned} \int_{-\omega}^{\infty} A^*(\omega+v)K(v)\,dv &= \sum_{-N}^{N} \alpha_n e^{i\gamma_n\omega} \int_{-\omega}^{\infty} K(v)e^{i\gamma_n v}dv \\ &= \sum_{-N}^{N} \alpha_n e^{i\gamma_n\omega} 2\pi k(\gamma_n) + o(1) \end{aligned}$$

as $\omega \to \infty$.

The desired inequality

$$\varliminf_{u\to\infty} A(u) \leq \varliminf_{\omega\to\infty} A^*_T(\omega)$$

now follows from (7), (8) and (9), in virtue of the definitions of $k(t) = k_T(t)$ and of $A^*_T(\omega)$.

THEOREM 2. *Suppose that the complex zeros of $\zeta(s)$ are all simple and lie on the line $\sigma = 1/2$, and let them be $(1/2) + i\gamma_n$ $(n = \pm 1, \pm 2, \cdots;$ $\gamma_{-n} = -\gamma_n;\ 0 < \gamma_1 < \gamma_2 < \cdots)$. Let α_n be the residue of $F(s)$ at $s = i\gamma_n$, where $F(s)$ denotes either of the functions*

$$\text{(i)}\ \frac{1}{((1/2)+s)\zeta((1/2)+s)}, \qquad \text{(ii)}\ \frac{\zeta(1+2s)}{((1/2)+s)\zeta((1/2)+s)}.$$

Then $\sum_1^{\infty} |\alpha_n|$ is divergent.

Consider the integral

$$I = \int (T + is)^k F(s)\, ds$$

taken in the positive sense round the rectangle with vertices $-\alpha, \beta, \beta + iT, -\alpha + iT$, indented upwards at the origin in case (ii), where α and β are fixed $(0 < \alpha < 1/4, 1/2 < \beta < 1)$, k is a fixed positive integer,[2] and $T \to \infty$ through a sequence T_m $(m = 1, 2, \cdots)$ such that

$$| 1/\zeta[(1/2) + s)] | < T^K \qquad (-1 \leq \sigma \leq 1, t = T = T_m),$$

where K is some positive constant. The existence of such a sequence T_m is well known (see, e. g., Titchmarsh [7], Theorem 18, p. 26).

By Cauchy's theorem of residues,

$$I = 2\pi i \sum_{0 < \gamma_n < T} (T - \gamma_n)^k \alpha_n = 2\pi i S_k(T),$$

say.

Let I_1, I_2, I_3, I_4 be the contributions to I of the four sides of the rectangle, numbered in the positive sense from the lower horizontal side. We have

$$I_1 = O(T^k),$$

$$\begin{aligned} I_2 &= \sum_{n=1}^{\infty} \frac{a_n}{n^{1/2}} \int_{\beta}^{\beta+iT} \frac{(T + is)^k ds}{((1/2) + s) n^s} \qquad [a_n = \mu(n) \text{ or } \lambda(n)] \\ &= \int_{\beta}^{\beta+iT} \frac{T^k ds}{(1/2) + s} + O(T^k) + O(\sum_{n=2}^{\infty} T^k / n^{1/2+\beta} \log n) \\ &= T^k \log T + O(T^k), \end{aligned}$$

by using the inequality $|(T + is)^k - T^k| \leq K_1 |s| T^{k-1}$ in the term $n = 1$, and deforming the path of integration into the broken line $\beta, T, T + iT, \beta + iT$ [or, alternatively, writing $n^{-s} ds = - d(n^{-s})/\log n$ and integrating by parts] in the terms $n \geq 2$.

Now the Riemann hypothesis (which we are assuming here) implies that, when $t \to \infty$, $\zeta(1 + 2s) = O(t^{\epsilon})$ uniformly for $-\alpha \leq \sigma \leq \beta$, and

$$\begin{aligned} \frac{1}{\zeta((1/2) + s)} &= \frac{2^{(1/2)-s} \pi^{-(1/2)-s} \cos[(1 + 2s)\pi/4] \Gamma((1/2) + s)}{\zeta((1/2) - s)} \\ &= O(t^{-\alpha+\epsilon}) \qquad (\sigma = -\alpha), \end{aligned}$$

where ϵ is an arbitrarily small positive number. (See, e. g., Titchmarsh [7], § 5. 13, (1), (2).) Hence

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[2] 'Integer' may be replaced by 'number' if we specify that the branch of $(T + is)^k$ which is real and positive for $s = it$, $t < T$, has to be taken.

$$I_3 = O(1 \cdot T^{-1} \cdot T^K \cdot T^\epsilon) = O(T^{K-1+\epsilon}),$$
$$I_4 = O(\int_0^T T^k (t+1)^{-1-\alpha+2\epsilon} dt) = O(T^k).$$

Collecting these resuts, we obtain

$$I = T^k \log T + O(T^k) + O(T^{K-1+\epsilon}),$$

so that, if $k > K - 1$,

$$2\pi i S_k(T) \sim T^k \log T \tag{10}$$

when $T \to \infty$ through the sequence T_m. This proves the theorem (and indeed more) since

$$| S_k(T) | \leq T^k \sum_{0<\gamma_n<T} | \alpha_n |.$$

3. Proof of Theorem A. We have

$$F(s) = \int_0^\infty A(u) e^{-su} du \qquad (\sigma > 1/2),$$

where $F(s)$ has either of the meanings assigned to it in Theorem 2, and $A(u)$ denotes the corresponding one of the functions

$$\text{(i)} \quad M(e^u) e^{-u/2}, \qquad \text{(ii)} \quad L(e^u) e^{-u/2}.$$

The conclusion of Theorem A is equivalent to

$$\overline{\lim_{u\to\infty}} A(u) = +\infty, \qquad \underline{\lim_{u\to\infty}} A(u) = -\infty. \tag{11}$$

Suppose that the first result (for example) is false. Then there is a constant A such that $A - A(u) \geq 0$ for $u \geq 0$, and the relation

$$\int_0^\infty \{A - A(u)\} e^{-su} du = (A/s) - F(s) = G(s)$$

(say), true in the first instance for $\sigma > 1/2$, holds for $\sigma > 0$ by Landau's theorem, since $G(s)$ is regular along the whole of the positive real axis $s > 0$. Hence $G(s)$, and therefore $F(s)$, is regular in the half-plane $\sigma > 0$; and also

$$| G(s) | \leq G(\sigma) = O(\sigma^{-1}) \quad \text{as} \quad \sigma \to +0,$$

so that $G(s)$, and therefore $F(s)$, can have no multiple pole on $\sigma = 0$. A similar argument with $A + A(u)$ applies if the second result (11) is false. This means (and all this is classical) that the results (11) are certainly true if $\zeta(s)$ has a complex zero off the line $\sigma = 1/2$, or if all the complex zeros lie on this line but include a multiple zero (and in these cases the hypothesis of linear independence of the γ_n is, of course, irrelevant).

We may, therefore, make the assumptions of Theorem 2. Define γ_n and

α_n as in Theorem 2 for $n = \pm 1, \pm 2, \cdots$, and as follows (in the two cases) for $n = 0$:

$$\text{(i)} \qquad \gamma_0 = 0, \qquad \alpha_0 = 0;$$
$$\text{(ii)} \qquad \gamma_0 = 0, \qquad \alpha_0 = 1/\zeta(1/2).$$

Then the conditions of Theorem 1 are satisfied for any given $T > 0$ if we choose N so that $\gamma_N \geq T$; and the conclusions (5) therefore hold.

But, if the γ_n $(0 < \gamma_n < T)$ are linearly independent,

$$\overline{\lim_{u\to\infty}} A^*_T(u) = \alpha_0 \pm 2 \sum_{0<\gamma_n<T} [1 - (\gamma_n/T)] \mid \alpha_n \mid, \tag{12}$$

since by Kronecker's theorem we can find arbitrarily large values of u for which the products $\gamma_n u$ $(0 < \gamma_n < T)$ are simultaneously as near as we please to $-\arg(\pm \alpha_n)$ (mod 2π). But the sum Σ on the right of (12) is

$$\geq \sum_{0<\gamma_n<T/2} \mid \alpha_n \mid /2$$

and is therefore arbitrarily large with T by Theorem 2. Hence

$$\overline{\lim_{u\to\infty}} A(u) \geq \lim_{T\to\infty} \overline{\lim_{u\to\infty}} A^*_T(u) = +\infty,$$

with a similar result for the lower limit.

The argument is not essentially affected if there are a finite number of relations of the type (3), since these will involve a last γ_n, say γ_M, and we can apply Kronecker's theorem to the γ_n in the range $\gamma_M < \gamma_n < T$.

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