LINNIK'S THEOREM
MATH 613E UBC

FINAL REPORT BY COLIN WEIR AND TOPIC PRESENTED BY NICK HARLAND

ABSTRACT. This report will describe in detail the proof of Linnik's theorem regarding the least prime in an arithmetic progression. Some background information is included, and references are given to fill in any omitted details. This report also surveys both the conjectured and computed results for Linnik's constant as well as a brief discussion of the more recent improvements.

1. INTRODUCTION

A classical theorem of Dirichlet says that the primes are evenly distributed amongst the equivalence class modulo q. This naturally leads to many questions about the distribution. For example, one may ask how large is the least prime congruent to a modulo q, denoted $p(a, q)$. We can first approach this question using the Prime Number Theorem for arithmetic progressions. Define

$$\theta(x; q, a) := \sum_{\substack{p \leq x \\ p \text{ prime} \\ p \equiv a \mod q}} \log p.$$

Then the theorem tells us that for $x < (\log q)^A$ there exists a c such that

$$\theta(x; q, a) = \frac{x}{\phi(q)} + O\left(x \exp(-c\sqrt{\log x})\right).$$

Notice that close to 0, $\theta(x; q, a) = 0$ and that as x increases $\theta(x; q, a)$ remains zero until x is greater than the first prime congruent to a modulo q. Thus if we can find a value of x such that $\theta(x; q, a) > 0$ then it must be the case that $p(a, q) \leq x$. We set $\theta(x; q, a) > 0$ and using a specific positive O-constant D we solve for x. We have that

$$\begin{aligned} Dx \exp\left(-c\sqrt{\log x}\right) &< \frac{x}{\phi(q)} \\ \sqrt{\log x} &< \frac{1}{c} \log(D\phi(q)). \end{aligned}$$

As $\phi(q) < q$, we may simplify the above to read

$$\begin{aligned}\sqrt{\log x} &< \frac{1}{c}\log(Dq)) \\ x &> \exp\left(\frac{1}{c^2}\log^2 Dq\right) \\ x &> (Dq)^{\frac{1}{c^2}\log Dq} \\ x &\gg q^{C\log q},\end{aligned}$$

for a computable constant C. So we obtain that $p(a,q) \ll q^{C\log q}$. Notice however that this bound is super-polynomial in q. However, if we assume the Riemann Hypothesis, then we can improve the error term in the Prime Number Theorem to

$$\theta(x;q,a) = \frac{x}{\phi(q)} + O\left(x^{1/2}\log^2 x)\right).$$

A similar calculation with this expression yields

$$p(a,q) \ll \phi(q)^2 \log^2 q.$$

So under the Riemann Hypothesis we get a result polynomial in q of order $q^{2+\epsilon}$. But an even stronger result of $q^{1+\epsilon}$ is conjectured. Therefore, before Linnik, the best unconditional lower bounds on $p(a,q)$ were extremely distant from the conditional ones. That was how the problem stood until Linnik's works [Lin44a] and [Lin44b] in 1944 where he proves

Linnik's Theorem: *There exists an effectively computable constant $L > 0$ such that*

$$\min\{p : p \text{ prime}, p \equiv a \bmod q\} =: p(a,q) \ll q^L.$$

Linnik himself never computed an explicit value for L but since his work many people have. An overview of these results is included in Table 1.

L	Name	Ref.
10000	Pan	[Pan57]
777	Chen	[J.65]
80	Jutila	[Jut77]
20	Graham	[Gra81]
16	Wang	[Wan86]
5.5	Heath-Brown	[HB92]
5.2	Xylouris	[Xyl09]

Table 1. Estimates For Linnik's Constant

In this paper we will prove the statement of Linnik's theorem as written above with no attempt to explicitly compute the constant. The next section contains some required background information and motivation for the approach of the proof. Section 3 contains a few key propositions, the proofs of which are covered in [IK04]. With these results we prove Linnik's Theorem in Section 4. We close with a brief discussion of some of the more recent improvements mainly due to Heath-Brown.

2. Preliminaries

The proof of Linnik's theorem relies on a few key ideas. First we need to derive an arithmetic function that has support on primes congruent to a modulo q. We will find this function while constructing a summatory function related to certain L-functions. We will eventually derive bounds from an investigation of the zeros of these L-functions. Finally, these bounds are manipulated to give bounds on the least prime in an arithmetic progression.

We begin by investigating the Dirichlet series of a character modulo q. Specifically, we consider

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

which we rewrite as an Euler product

$$L(s,\chi) = \prod_{p \text{ prime}} (1-\chi(p)p^{-s})^{-1}.$$

The Euler product and Dirichlet series can be shown to converge absolutely for $\Re(s) > 1$. Using the Euler product, we derive a formula for the logarithmic derivative of $L(s,\chi)$. Taking the logarithm, we see that

$$\begin{aligned}\log\left(L(s,\chi)\right) &= \log\left(\prod_{p \text{ prime}} (1-\chi(p)p^{-s})^{-1}\right) \\ &= \sum_{p \text{ prime}} -\log\left((1-\chi(p)p^{-s})\right).\end{aligned}$$

Using the Taylor Expansion of log,

$$\begin{aligned}-\log\left(L(s,\chi)\right) &= \sum_{p \text{ prime}} \log\left((1-\chi(p)p^{-s})\right) \\ &= \sum_{p \text{ prime}} \sum_{k} \frac{1}{k}\chi(p)^k p^{-ks}\end{aligned}$$

and since the series is absolutely convergent, differentiating gives

$$\begin{aligned}-\frac{L'(s,\chi)}{L(s,\chi)} &= \sum_{p \text{ prime}} \sum_{k} \left(p^k\right)^{-s} \chi(p^k)\log(p) \\ &= \sum_{n} \Lambda(n)\chi(n)n^{-s}\end{aligned}$$

where Λ is defined to be the function that makes the above equality true; specifically

$$\Lambda(n) = \begin{cases} \log(p) & n = p^k \text{ for some } k \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

The Mellin Inversion formula provides an expression for the sum of $\Lambda(n)$ in terms of L-functions;

$$\sum_{n\leq x}\Lambda(n)\chi(n) = -\lim_{T\to\infty}\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{L'}{L}(s,\chi)\frac{x^s}{s}ds.$$

Next, we use Perron's Formula to approximate the integral above up to a bound T and then move the line of integration to arrive at the following expression:

$$\psi(x,\chi) := \sum_{n\leq x}\Lambda(n)\chi(n) = \delta_\chi x - \sum_{|\mathrm{Im}(\rho)|\leq T}\frac{x^\rho}{\rho} + O\left(\frac{x\left(\log(xq)\right)^2}{T}\right) \tag{1}$$

where ρ runs over the zeros of $L(s,\chi)$, and $\delta_\chi = 0$ unless χ is the principal character in which case $\delta_\chi = 1$ (see [IK04, Prop. 5.25]).

To demonstrate the idea behind the proof of Linnik's Theorem, let's take a closer look at the function $\psi(x,\chi)$ when χ is the principal character, which we will denote χ_0. In this case $L(s,\chi_0)$ is just the Riemann Zeta function $\zeta(s)$. Similar to $\theta(x,q;a)$, for values of x near 0, $\psi(x,\chi_0) = 0$. Furthermore, as x increases, $\psi(x,\chi)$ remains zero until we reach the first prime power 2, where it takes the value $\log(2)$. So if we were looking for a bound on how big the smallest prime power is (potentially with the power 1), one way to do it is to prove that $\psi(x,\chi_0) > 0$ for all x greater than some n. With only a little more work we can describe how small the smallest prime is. That is, if one knows something about the zeros of $\zeta(s)$ they could describe the behavior of $\psi(x,\chi_0)$ and consequently the growth of primes. This idea from the proof of the Prime Number Theorem is essentially the idea behind the proof of Linnik's theorem as well. However, we don't want to examine the function $\psi(x,\chi)$, which increments at all prime powers, but a different function that only increments on prime powers congruent to a modulo q. With this in mind, we define the following function:

$$\psi(x;q,a) := \frac{1}{\phi(q)}\sum_{\chi \bmod q}\bar{\chi}(a)\psi(x,\chi).$$

Using the orthogonality of characters we see that

$$\begin{aligned}
\psi(x;q,a) &= \frac{1}{\phi(q)}\sum_{\chi \bmod q}\bar{\chi}(a)\psi(x,\chi)\\
&= \frac{1}{\phi(q)}\sum_{\chi \bmod q}\bar{\chi}(a)\sum_{n\leq x}\Lambda(n)\chi(n)\\
&= \frac{1}{\phi(q)}\sum_{n\leq x}\Lambda(n)\sum_{\chi \bmod q}\chi(n)\bar{\chi}(a)\\
&= \frac{1}{\phi(q)}\sum_{n\leq x}\Lambda(n)\sum_{\chi \bmod q}\chi(na^{-1})\\
&= \frac{1}{\phi(q)}\sum_{\substack{n\leq x\\ n\equiv a \bmod q}}\Lambda(n).
\end{aligned}$$

Notice that $\psi(x;q,a)$ is positive for all x greater than the first prime congruent to a modulo q. Furthermore, by substituting equation (1) into the definition of $\psi(x;q,a)$ we obtain the *explicit formula*

$$\psi(x;q,a) = \frac{x}{\phi(q)} - \frac{1}{\phi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) \sum_{|\mathrm{Im}\rho| \leq T} \frac{x^\rho}{\rho} + O\left(\frac{x\left(\log(xq)\right)^2}{T}\right). \qquad (2)$$

Hence we have an expression for $\psi(x;q,a)$ in terms of the zeros of L-functions. So if we can describe the behavior of the zeros of $L(s,\chi)$ for all characters χ modulo q, then we can describe the number of prime powers congruent to a modulo q. With this information it is a relatively simple matter to describe all primes congruent to a modulo q.

3. Required Results

The proof of Linnik's Theorem is in essence an effort to describe the growth of $\psi(x;q,a)$ via the explicit formula. As such, we need to make use of results about the distribution and density of the zeros of $L(s,\chi)$ for all characters χ. We therefore turn our attention to their product. We denote

$$L_q(s) := \prod_{\chi \bmod q} L(s,\chi).$$

For $1/2 \leq \alpha \leq 1$ and $T \geq 1$ we define $N(\alpha,T,\chi)$ to be the number of zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ counted with multiplicity in the rectangle $\alpha < \sigma \leq 1$, $|t| \leq T$. And we write

$$N_q(\alpha,T) := \sum_{\chi \bmod q} N(\alpha,T,\chi)$$

for the number of zeros of $L_q(s)$ counted with multiplicity in the rectangle.

Following the approach of [IK04] we will make use of the following three results.

Theorem 3.1 - The Zero-Free Region: *Let $\sigma := \Re(s)$. There is a positive constant c_1, effectively computable, such that $L_q(s)$ has at most one zero in the region*

$$\sigma \leq 1 - c_1/\log(qT),$$

If such a zero exists, then it is simple and the zero of $L(s,\chi_1)$ where χ_1 is real, and non-principal.

Theorem 3.2 - The Log-Free Density Theorem: *There are effectively computable constants $c_1, c_2 > 0$ such that for any $1/2 < \alpha < 1$, $T \geq 1$,*

$$N_q(\alpha,T) \leq c(qT)^{c_2(1-\alpha)}$$

Theorem 3.3 - The Deuring-Heilbron Phenomenon: *There is a constant c_3 such that if there does exist an exceptional zero $\rho_1 = \beta_1$ with $L(\rho,\chi_1) = 0$ and*

$1 - c_1/\log(qT) \leq \beta_1 \leq 1$, *then for all characters* χ *modulo* q, $L(s,\chi)$ *has no other zeros in the region*

$$\sigma \geq 1 - c_3 \frac{|\log(1-\beta_1)\log(qT)|}{\log(qT)}, \quad |t| \leq T.$$

Notice that Theorem 3.3 is about how the zeros of *all* characters modulo q are repelled by the presence of an exceptional zero of *one* character. The proofs of these theorems will not be discussed in this paper. However we note that the zero-free region is due to Landau and the proof can be found in [IK04, Thm 5.26]. Theorems 3.2 and 3.3 are due to Linnik in [Lin44a] and [Lin44b] respectively, however the proofs of these results are also the subjects of Sections 18.2 and 18.3 in [IK04].

4. Proof of Linnik's Theorem

We present the proof as demonstrated by Nick Harland. He followed the approach used in [IK04], which borrows heavily from [Gra81].

Recall that we aim to prove the following.

Linnik's Theorem: *There exists an effectively computable constant* $L > 0$ *such that*

$$\min\{p : p \text{ prime}, p \equiv a \bmod q\} =: p(a,q) \ll q^L.$$

We begin by establishing some notation. Let $c, c_1, c_2, c_3, c_4, \rho, \beta_1$ be as in Theorems 3.1,3.2, and 3.3. Assume without loss of generality that $0 < c_1, c_3 < 1$ and $c, c_2 > 1$. To keep the calculations clean, we set $R = x^{1/(2c_2)}$.

We will prove Linnik's theorem through a series of propositions regarding the behavior of $\psi(x;q,a)$, beginning with the following.

Proposition 4.1 *Let* $x \geq q^{4c_2}$. *Then*

$$\psi(x;q,a) = \frac{x}{\phi(q)}\left(1 - \chi_1(a)\frac{x^{\beta_1-1}}{\beta_1} + \theta c x^{\eta/2} + O\left(\frac{\log q}{q}\right)\right)$$

where θ *is some function with* $|\theta| < 4$, *and*

$$\eta := \begin{cases} \frac{c_3|\log(2(1-\beta_1)\log q)|}{2\log q} & \text{if } \beta_1 \text{ exists} \\ \frac{c_1}{2\log q} & \text{otherwise} \end{cases}$$

Proof. We begin with the explicit formula for primes in arithmetic progression; Equation 2.

$$\psi(x;q,a) = \frac{x}{\phi(q)} - \frac{1}{\phi(q)}\sum_{\chi \bmod q}\bar{\chi}(a)\sum_{|\mathrm{Im}\rho|\leq R}\frac{x^\rho}{\rho} + O\left(\frac{x(\log(xq))^2}{T}\right).$$

Let $0 < T \leq R$ and consider the magnitude of a portion of the summation above:

$$\left|\sum_{\chi} \bar{\chi}(a) \sum_{\frac{1}{2}T\leq|\gamma|\leq T} \frac{x^{\rho}}{\rho}\right| \leq \sum_{\chi} \sum_{|\gamma|\leq T} \frac{x^{\beta}}{|\rho|} \leq \sum_{\chi} \sum_{|\gamma|\leq T} \frac{x^{\beta}}{T/2}$$

by bringing the absolute values inside and bounding ρ. Notice that this sum can be represented by a Riemann-Stieltjes integral. Namely,

$$\begin{aligned}\sum_{\chi} \sum_{|\gamma|\leq T} \frac{x^{\beta}}{T/2} &= -\frac{2}{T}\int_{\frac{1}{2}}^{1} x^{\alpha} d\left(N_q(\alpha, T)\right) \\ &= -\frac{2}{T} x N_q(1,T) + x^{\frac{1}{2}}\frac{2}{T}N_q\left(\frac{1}{2}, T\right) + -\frac{2}{T}\int_{\frac{1}{2}}^{1} x^{\alpha}\log(x) N_q(\alpha, T) d\alpha.\end{aligned}$$

Now we substitute in the bound on $N_q(\alpha, T)$ from Theorem 3.1, giving that the above is at most

$$\begin{aligned}&-\frac{2}{T}c(qT)^{\frac{1}{2}c_2}x^{\frac{1}{2}} + \frac{2c}{T}x\log x\int_{\frac{1}{2}}^{1}\left(\frac{(qT)^{c_2}}{x}\right)^{1-\alpha} d\alpha \\ &\leq -\frac{2}{T}c(qT)^{\frac{1}{2}c_2}x^{\frac{1}{2}} + \frac{2c}{T}x\log x\frac{1}{\log\left(\frac{x}{(qT)^{c_2}}\right)}\left[\left(\frac{(qT)^{c_2}}{x}\right)^{0} - \left(\frac{(qT)^{c_2}}{x}\right)^{\frac{1}{2}}\right].\end{aligned}$$

Noticing that,

$$\begin{aligned}&\frac{2c}{T}(qT)^{\frac{1}{2}c_2}x^{\frac{1}{2}} - \frac{2c}{T}\frac{x\log x}{\log\left(\frac{x}{(qT)^{c_2}}\right)}\left(\frac{(qT)^{c_2}}{x}\right)^{\frac{1}{2}} \\ &= \frac{2c}{T}(qT)^{\frac{1}{2}c_2}x^{\frac{1}{2}}\left(1 - \frac{x^{1/2}\log x}{\log\left(\frac{x}{(qT)^{c_2}}\right)}\right) \\ &< \frac{2c}{T}(qT)^{\frac{1}{2}c_2}x^{\frac{1}{2}}\left(1 - \frac{x^{1/2}\log x}{\log x}\right) \\ &< \frac{2c}{T}(qT)^{\frac{1}{2}c_2}x^{\frac{1}{2}}\left(1 - x^{1/2}\right) < 0\end{aligned}$$

we obtain that

$$\left|\sum_{\chi} \bar{\chi}(a) \sum_{\frac{1}{2}T\leq|\gamma|\leq T} \frac{x^{\rho}}{\rho}\right| \leq \frac{2c}{T}x\log x\frac{1}{\log\left(\frac{x}{(qT)^{c_2}}\right)}.$$

Finally, using $x \geq (qT)^{c_2+1/3}$,

$$\left|\sum_{\chi} \bar{\chi}(a) \sum_{\frac{1}{2}T\leq|\gamma|\leq T} \frac{x^{\rho}}{\rho}\right| \leq \frac{2cx}{T}\frac{1}{1 - c_2\frac{\log qT}{\log x}} \leq 6c(c_2 + 1/3)\frac{x}{T}. \tag{3}$$

Therefore, looking back at the explicit formula,

$$\psi(x;q,a) = \frac{x}{\phi(q)} - \frac{1}{\phi(q)} + \sum_{\chi} \bar{\chi}(a) \sum_{|\gamma| \le T} \frac{x^\rho}{\rho} + O\left(\frac{x \log x}{R} + \begin{array}{c}\text{error of summing only}\\ \text{to T instead of R}\end{array} \right).$$

We can bound the error term using equation 3. In particular, equation 3 give us the bound on the error for each time we reduce our sum from R to $\frac{1}{2}R$. Repeating, we have a bound on the error from reducing the sum from $1/2R$ to $1/4R$, and then to $1/8R$, etc., until we reach T. Thus, taking an infinite sum, the total error is bounded by

$$\sum_{i=1}^{\infty} \frac{x}{R2^{i-1}} = O\left(\sum_{i=1}^{\infty} \frac{x}{T2^{i-1}} \right).$$

Consequently,

$$\begin{aligned}
\psi(x;q,a) &= \frac{x}{\phi(q)} - \frac{1}{\phi(q)} + \sum_{\chi} \bar{\chi}(a) \sum_{|\gamma| \le T} \frac{x^\rho}{\rho} + O\left(\frac{x \log x}{R} + \frac{1}{T\phi(q)} \sum_{i=1}^{\infty} \frac{x}{2^{i-1}} \right) \\
&= \frac{x}{\phi(x)} - \frac{1}{\phi(q)} + \sum_{\chi} \bar{\chi}(a) \sum_{|\gamma| \le T} \frac{x^\rho}{\rho} + O\left(\frac{x \log x}{R} + \frac{1}{T\phi(q)} \right)
\end{aligned}$$

and if $T = q$ then

$$\psi(x;q,a) = \frac{x}{\phi(q)} - \frac{1}{\phi(q)} + \sum_{\chi} \bar{\chi}(a) \sum_{|\gamma| \le T} \frac{x^\rho}{\rho} + O\left(\frac{x \log q}{q\phi(q)} \right).$$

Now that we have the same error term as the statement of the proposition, we turn our attention to the sum up to T over the zeros of the Dirichlet characters. If an exceptional zero exists, we remove it from the sum. Then, using the values of η given in the statement of the proposition,

$$\begin{aligned}
\left| \sum_{\chi} \bar{\chi}(a) \sum_{\substack{|\gamma| \le T \\ \rho \text{ unexceptional}}} \frac{x^\rho}{\rho} \right| &\le \sum_{\chi} \bar{\chi}(a) \sum_{\substack{|\gamma| \le T \\ \rho \text{ unexceptional}}} x^\beta \\
&= -2 \int_{\frac{1}{2}}^{1-\eta} x^\alpha dN_q(\alpha, T) \\
&= 2x^{\frac{1}{2}} N_q(1/2, T) + 2(\log x) \int_{\frac{1}{2}}^{1-\eta} x^\alpha N_q(\alpha, T) d\alpha.
\end{aligned}$$

Appealing to Theorem 3.2 and making a similar calculation to the one above, we arrive at the bound

$$\begin{aligned}
2x^{\frac{1}{2}}N_q(1/2,T) + 2(\log x)\int_{\frac{1}{2}}^{1-\eta} x^{\alpha}N_q(\alpha,T)d\alpha & \\
&\leq 2cq^{c_2}x^{\frac{1}{2}} + 2cx(\log x)\int_{\frac{1}{2}}^{1-\eta} x^{\alpha}\left(\frac{q^{2c_2}}{x}\right)^{1-\alpha} d\alpha \\
&\leq \frac{2cx(\log x)}{\log xq^{-2c_2}}\left(\frac{q^{2c_2}}{x}\right)^{\eta} \\
&\leq 4cx^{1-\eta/2},
\end{aligned}$$

which gives the desired bound on $\psi(x;q,a)$. $\square$

We have established an estimate for $\psi(x;q,a)$ to work from. The last step in the proof is to use the estimate from Proposition 4.1 to create lower bounds for $\psi(x;q,a)$. To do that we proceed in two different ways depending on whether or not an exceptional zero exists. The simpler situation is when an exceptional zero does not exist. In that case we have the following result.

Proposition 4.2 *If β_1 does not exist, then if $x \geq q^{4c_2}$ then*

$$\psi(x;q,a) = \frac{x}{\phi(q)}\left(1 + \theta c \exp\left(\frac{-c_1 \log x}{4 \log q}\right)\left(\frac{\log q}{q}\right)\right).$$

Proof. Set $\eta = \frac{1}{2}\log q$ in Proposition 4.1. $\square$

In the more difficult case where an exceptional zero does exist we have the following.

Proposition 4.3 *Suppose β_1 exists, and suppose that $\delta_1 := 1 - \beta_1 \leq \frac{c_1}{2\log q}$. Set $\nu := \max\left\{4c_2, \frac{4}{c_1}, \frac{4\log(8c)}{c_3|\log c_1|}\right\}$. If $x \geq q^{\nu}$, then*

$$\psi(x;q,a) \geq \frac{x}{\phi(q)}\frac{\delta_1 \log q}{3c_1}\left(1 + O\left(q^{-\frac{1}{2}}\right)\right).$$

Proof. We consider two pieces of the expression for $\psi(x;q,a)$ in Proposition 4.1. First, using the assumptions on x and δ_1 we have

$$1 - \chi_1(a)\frac{x^{\beta_1 - 1}}{\beta_1} \geq 1 - \frac{x^{-\delta_1}}{\beta_1} \geq \beta_1 - x^{-\delta_1} \geq \beta_1 - q^{-\nu\delta_1} = 1 - \delta_1 - q^{-\nu\delta_1}.$$

Using that $1 - e^{-x} \geq x/(x+1)$, which can be proved via Taylor series, we get that the above is greater than

$$\frac{\nu\delta_1 \log q}{1 + \nu\delta_1 \log q} - \delta_1 \geq \frac{\nu\delta_1 \log q}{1 + \nu c_1/2} - \delta_1 = \frac{\delta_1 \log q}{1/\nu + c_1/2} - \delta_1.$$

Finally, substituting in for ν we get that the above is greater than

$$\frac{\delta_1 \log q}{c_4/4 + c_1/2} - \delta_1 = \frac{4\delta_1 \log q}{3c_1} - \delta_1.$$

Secondly,

$$x^{\nu/2} \le q^{-\nu/2} \le q^{\frac{-c_3 \log(2\delta_1 \log q)|}{4 \log q}} \le q^{-\frac{\log(8c)|\log(2\delta_1 \log q)|}{|\log c_1| \log q}}.$$

Recalling that $c_1 > 1$, we can can drop the absolute values around $\log c_1$. Hence, the above is equal to

$$q^{-\frac{\log(8c)|\log(2\delta_1 \log q)|}{\log c_1 \log q}} = e^{-\frac{\log(8c)|\log(2\delta_1 \log q)|}{\log c_1}} = 2\delta_1 \log q (2\delta_1 \log q)^{-\frac{\log 8c}{\log c_1} - 1}.$$

Substituting for δ_1 yields

$$\begin{aligned} 2\delta_1 \log q (2\delta_1 \log q)^{-\frac{\log 8c}{\log c_1} - 1} &\le \left(\frac{2\delta_1 \log q}{c_1} \right) c_1^{-\frac{\log 8c}{\log c_1} - 1} \\ &\le \frac{2\delta_1 \log q}{c_1 8c} \\ &= \frac{\delta_1 \log q}{4cc_1}. \end{aligned}$$

Using these inequalities in Proposition 4.1 we find

$$\begin{aligned} \psi(x; q, a) &\ge \frac{x}{\phi(q)} \left(\frac{4\delta_1 \log q}{3c_1} - \delta_1 - \frac{\delta_1 \log q}{c_1} + O\left(\frac{\log q}{q} \right) \right) \\ &= \frac{x}{\phi(q)} \left(\frac{\delta_1 \log q}{3c_1} - \delta_1 - O\left(\frac{\log q}{q} \right) \right). \end{aligned}$$

Factoring out $\delta_1 \log q/(3c_1)$ gives the result. $\square$

As $\delta_1 \log q \gg q^{-\frac{1}{2}}$ we can combine the results of Propositions 4.3 and 4.2 to conclude

$$\psi(x; q, a) \gg \frac{x}{\phi(q) q^{\frac{1}{2}}} \gg 1$$

for $x \ge q^{\nu}$.

Recall that $\psi(x; q, a)$ increments at primes and prime powers. We would like to infer a bound on

$$\theta(x; q, a) := \sum_{\substack{p \le x \\ p \text{ prime} \\ p \equiv a \mod q}} \log(p),$$

which increments only at primes. Comparing $\psi(x;q,a)$ and $\theta(x;q,a)$ we find that

$$\psi(x;q,a)-\theta(x;q,a)=\sum_{i=2}^{\infty}\sum_{\substack{p^i\le x\\ p\text{ prime}\\ p\equiv a \mod q}}\log(p)<\sum_{i=2}^{\infty}\sum_{\substack{p^i\le x\\ p\text{ prime}}}\log(p).$$

As such, we define $\theta(x):=\sum_{\substack{p<x\\ p\text{ prime}}}\log p$ and obtain

$$\psi(x;q,a)-\theta(x;q,a)<\sum_{i=2}^{\infty}\theta(x^{\frac{1}{n}})=\sum_{i=2}^{log_2(x)}\theta(x^{\frac{1}{n}})$$

as $\theta(x)=0$ for $x<2$. Hence, to relate $\psi(x;q,a)$ and $\theta(x;q,a)$ we need to investigate the growth of $\theta(x^1/n)$.

Using Riemann-Stieltjes integration we see that

$$\theta(x^{1/n})=\int_1^{x^{1/n}}\log y\,d(\pi(y))=\pi(x^{1/2})\log(x^{1/2})-\int_1^{x^{1/n}}\frac{\pi(y)}{y}dy.$$

Using the trivial bound $\pi(y)>1$ we can simplify the integral yielding

$$\theta(x^{1/n})<\pi(x^{1/n})\log(x^{1/n})-\int_1^{x^{1/n}}\frac{1}{y}dy=\big(\pi(x^{1/n})-1\big)\log(x^{1/n}).$$

And so by the Prime Number Theorem we see that $\theta(x^{1/n})\ll x^{1/n}$. Therefore,

$$\psi(x;q,a)-\theta(x;q,a)\ll\sum_{n=2}^{\infty}x^{1/n}\ll x^{1/2}.$$

Finally, we see that

$$\theta(x;q,a)=\psi(x;q,a)+O(x^{1/2})\gg\frac{x}{q^{1/2}\phi(q)}+Cx^{1/2}\gg\frac{x}{q^{1/2}\phi(q)}$$

for $x\gg q^L$ for some explicitly computable number L. Recalling that $\theta(x;q,a)$ is zero until x is greater than a prime congruent to a modulo q, we conclude that $p(a,q)\ll q^L$, where L is a effectively computable constant. Linnik's Theorem is proved.

5. Concluding Remarks

One could work through the proof we have provided and compute a numerical value for L. This was first done by Pan [Pan57] who achieved a value of $L=10000$ and later more carefully $L=5448$. The more recent result of Heath-Brown achieves $L=5.5$ using essentially similar but considerably better methods. To get a sense of his improvements, we note that he is able to use $c_1=0.348$, which would only give us a value of $\nu>11.5$. We conclude this report with a brief discussion on the types of improvements that Heath-Brown made.

The proof we provided was an investigation of $\psi(x;q,a)$; essentially a search for a value of x to which $\psi(x;q,a)>0$. Besides making use of modern improvements of Theorems 3.1, 3.2 and 3.3, Heath-Brown investigates a slightly different function. He defines

$$\Sigma := \sum_{n=1}^{\infty} \Lambda(n)\chi(n)n^{-s} f\left(\frac{\log n}{\log q}\right),$$

where $f(x)$ is a continuous function with support on $[0,x_0)$. He similarly finds an *explicit formula* for Σ involving L-functions and also looks for bounds based on their zeros.

Heath-Brown also argues using a "zero free region," but with at most four zeros in it. He credits the idea for this improvement to Graham, who in [Gra81] proves the following.

Theorem 5.1 *If q is sufficiently large, then $L_q(s) = \prod_{\chi \bmod q} L(s,\chi)$ has at most two distinct zeros $\rho=\sigma+it$ in the region*

$$\sigma \geq 1 - \frac{0.2069}{\log q(2+|t|)},$$

and at most four zeros in the region

$$\sigma \geq 1 - \frac{0.2769}{\log q(2+|t|)}.$$

However, using his new explicit formula with the function f, Heath-Brown improves Graham's bounds and proves the following three theorems.

Theorem 5.2 *If q is sufficiently large, then $L_q(s)$ has at most one zero in the region*

$$\sigma \geq 1 - \frac{0.348}{\log q}, \quad |t| \leq 1.$$

Such a zero, if it exists, is real and simple, and corresponds to a non-principal real character.

Theorem 5.3 *If q is sufficiently large, then $L_q(s)$ has at most two zeros, counted with multiplicity, in the region*

$$\sigma \geq 1 - \frac{0.696}{\log q}, \quad |t| \leq 1.$$

Moreover, for large enough q, there exists a character $\chi_1 \bmod q$ such that $L(s,\chi)$ is non-vanishing for

$$\sigma \geq 1 - \frac{0.702}{\log q}, \quad |t| \leq 1,$$

for all characters $\chi \bmod q$ with $\chi \neq \chi_1, \overline{\chi_1}$.

Theorem 5.4 *If q is sufficiently large, there exist characters $\chi_1, \chi_2 \mod q$ such that $L(s,\chi)$ is non-vanishing for*

$$\sigma \geq 1 - \frac{0.857}{\log q}, \quad |t| \leq 1.$$

for all characters $\chi \mod q$ with $\chi \neq \chi_1, \overline{\chi_1}, \chi_2, \overline{\chi_2}$.

Notice that these three theorems encapsulate statements about both zero-free regions and Deuring-Heilbronn phenomena. In fact, Heath-Brown argues using variational calculus that the function f was chosen optimally to maximize the bounds given in these theorems.

One more improvement comes from taking a closer look at the zeros of individual L-functions. Specifically, Heath-Brown counts the characters χ whose L-function has a zero in the region $1 - \lambda/\log q$. He denotes this $N(\lambda)$ and shows

$$N(\lambda) \leq (1+\epsilon)\frac{67}{6\lambda}\left(e^{\frac{73\lambda}{30}} - e^{\frac{16\lambda}{15}}\right)$$

for $\lambda \leq 1/3 \log\log\log q$. He credits this as following directly from a bound on character sums given by Burgess [Bur86]

Heath-Brown's value of $L = 5.5$ also relies heavily on numerical computations; and admittedly Heath-Brown made no effort to compute these values to the finest precision. Indeed, he closes his paper with a list of nine small technical improvements that could be made to improve his result. This was essentially what gave Xylooris a value of $L = 5.2$ in [Xyl09].

References

[Bur86] D. A. Burgess, *The character sum estimate with $r = 3$*, J. London Math. Soc. (2) **33** (1986), no. 2, 219–226.

[Gra81] S. Graham, *On Linnik's constant*, Acta Arith. **39** (1981), no. 2, 163–179.

[HB92] D. R. Heath-Brown, *Zero-free regions for Dirichlet L-functions, and the least prime in an arithmetic progression*, Proc. London Math. Soc. (3) **64** (1992), no. 2, 265–338.

[IK04] H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004.

[J.65] Chen J., *On the least prime in an arithmetical progression*, Sci. Sinica **14** (1965), 1868–1871.

[Jut77] Matti Jutila, *On Linnik's constant*, Math. Scand. **41** (1977), no. 1, 45–62.

[Lin44a] U. V. Linnik, *On the least prime in an arithmetic progression. I. The basic theorem*, Rec. Math. [Mat. Sbornik] N.S. **15(57)** (1944), 139–178.

[Lin44b] ———, *On the least prime in an arithmetic progression. II. The Deuring-Heilbronn phenomenon*, Rec. Math. [Mat. Sbornik] N.S. **15(57)** (1944), 347–368.

[Pan57] C. D. Pan, *On the least prime in an arithmetical progression*, Sci. Record (N.S.) **1** (1957), 311–313.

[Wan86] W. Wang, *On the least prime in an arithmetic progression*, Acta Math. Sinica **29** (1986), no. 6, 826–836.

[Xyl09] T. Xylouris, *On linnik's constant*, arXiv:0906.2749, 2009.

E-mail address: cjweir@ucalgary.ca