Zeros on the Critical Line

Eric Naslund

April 8, 2011

Abstract

The purpose of this report is to exhibit the proofs of two major results regarding the zeros of ζ on the critical line. First, we present a proof of Hardy's 1914 result, namely that there are infinitely many zeros of ζ on the critical line. Next we show Selbergs proof that the proportion of zeros of ζ on the critical line is positive.

March 14 2011

Part I
Hardy's Result

1 Introduction

We begin with some basic definitions. For $T > 0$, if there are no zeros of $\zeta(s)$ with imaginary part equal to T let

$$N(T) = |\{\beta + i\gamma : \ \zeta(\beta + i\gamma) = 0, \ 0 < \beta < 1, \ 0 < \gamma < T\}|$$

and if $\zeta(s)$ has a zero with imaginary part T let

$$N(T) = \frac{N(T^+) + N(T^-)}{2}.$$

This is the zero counting function for $\zeta(s)$, and we can show that (Corollary 14.3 of [2])

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$

Since it is believed that all the zeros of $\zeta(s)$ lie on the line $\beta = \frac{1}{2}$, it is natural to consider the related function

$$N_0(T) = \left| \left\{ \beta + i\gamma : \ \zeta(\beta + i\gamma) = 0, \ \beta = \frac{1}{2}, \ 0 < \gamma < T \right\} \right|$$

which is defined with similar considerations as above when T is the ordinate of a zero of $\zeta(s)$. $N_0(T)$ counts the zeros on the critical line, and we see that upon assuming the Riemann Hypothesis we must have

$$N_0(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T).$$

(This can actually be improved to $N_0(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O\left(\frac{\log T}{\log\log T}\right)$ when the Riemann Hypothesis is assumed.)

Our goal is to examine some of the major results regarding lower bounds on the size of $N_0(T)$. We will make use of the familiar function ξ which is defined by

$$\xi(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}. \tag{1.1}$$

ξ satisfies the functional equation $\xi(s) = \xi(1-s)$ (Corollary 10.3 of [2]) and hence is real on the line $\sigma = \frac{1}{2}$. Most importantly, notice that inside the critical strip, $\xi(\beta + i\gamma) = 0$ if and only if $\zeta(\beta + i\gamma) = 0$, so we may focus our attention on the zeros of ξ. Since we are trying to count zeros on only the critical line it is natural to introduce the single variable function $\Xi: \mathbb{R} \to \mathbb{R}$ defined by

$$\Xi(t) = \xi\left(\frac{1}{2} + it\right).$$

Again, the zeros of Ξ correspond exactly to the zeros of ζ on the critical line.

1.1 Brief History of Current Results

In 1914, Hardy showed that ζ has infinitely many zeros on the critical line, $\sigma = \frac{1}{2}$. In 1921 Hardy and Littlewood showed that $N_0(T) \gg T$. Later, in 1942, Selberg proved that $N_0(T) \gg T\log T$, and hence that a positive proportion of the zeros lie on the critical line. In 1974, Levinson showed that the proportion is at least $\frac{1}{3}$, and in 1989, Conrey increased this to $\frac{2}{5}$ by using Levinsons method.

2 Preliminaries

Let

$$\psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}.$$

The function $\psi(x)$ will play a major role in the proofs regarding the zeros of the zeta function. This is because $\frac{1}{s(2s-1)}\xi(2s) = \zeta(2s)\Gamma(s)\pi^{-s}$ is the Mellin transform of $\psi(x)$.

Proposition 1. *For $\sigma > \frac{1}{2}$ we have the identity*

$$\zeta(2s)\Gamma(s)\pi^{-s} = \int_0^{\infty} x^s\psi(x)\frac{dx}{x}.$$

Proof. By Euler's formula for the Gamma function we have

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt.$$

Making the substitution $t = n^2 \pi x$ we find

$$\Gamma(s)\pi^{-s} n^{-2s} = \int_0^\infty e^{-n^2\pi x} x^{s-1} dx.$$

Hence if $\sigma > \frac{1}{2}$, summing over n and switching the order of the sum and the integral yields

$$\Gamma(s)\pi^{-s} n^{-2s} = \int_0^\infty \psi(x) x^{s-1} dx$$

as desired. □

Corollary 2. *The function* $\zeta(2s)\Gamma(s)\pi^{-s}$ is the Mellin transform of $\psi(x)$. Consequently for $\sigma > \frac{1}{2}$ we have the inverse transform

$$\psi(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \zeta(2s)\Gamma(s)\pi^{-s} x^{-s} ds,$$

or equivalently for $\sigma > 1$ we have

$$\psi(y) = \frac{1}{4\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} y^{-\frac{s}{2}} ds. \tag{2.1}$$

The following functional equation for $\psi(x)$ we be used throughout the proof of Hardy's result.

Lemma 3. $\psi(x)$ *obeys the functional equation*

$$2\psi(x) + 1 = x^{-\frac{1}{2}}\left(2\psi\left(\frac{1}{x}\right) + 1\right). \tag{2.2}$$

Proof. This follows from the functional equation for the Jacobi theta function

$$\theta(x) = \sum_{-\infty}^{\infty} e^{-n^2\pi x}.$$

It is well known that

$$\theta(x) = \frac{1}{\sqrt{x}}\theta\left(\frac{1}{x}\right),$$

and this also follows from the Poisson summation formula. Then, since $2\psi(x) + 1 = \theta(x)$ we see that

$$2\psi(x) + 1 = x^{-\frac{1}{2}}\left(2\psi\left(\frac{1}{x}\right) + 1\right)$$

as desired. □

Proposition 4. *For all $s \in \mathbb{C}\backslash\{0,1\}$ we have*

$$\zeta(s)\Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s} = \frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-\frac{1}{2}}\right)\psi(x)dx.$$

Proof. By 1 we have

$$\psi(x) = \frac{x^{-\frac{1}{2}}\left(2\psi\left(\frac{1}{x}\right)+1\right)-1}{2}$$

$$\zeta(s)\Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s} = \int_1^\infty \frac{x^{\frac{1}{2}s}\psi(x)}{x}dx + \int_0^1 \frac{x^{\frac{1}{2}s}\psi(x)}{x}dx. \tag{2.3}$$

Then by 3 the second integral becomes

$$\frac{1}{2}\int_0^1 \frac{x^{\frac{1}{2}s}x^{-\frac{1}{2}}\left(2\psi\left(\frac{1}{x}\right)+1\right)}{x} - x^{\frac{1}{2}s-1}dx = \int_0^1 \frac{x^{\frac{1}{2}s-\frac{1}{2}}\psi\left(\frac{1}{x}\right)}{x} + \frac{1}{2}\int_0^1 x^{\frac{1}{2}s-\frac{3}{2}} - x^{\frac{1}{2}s-1}dx$$

$$= \int_0^1 \frac{x^{\frac{1}{2}s-\frac{1}{2}}\psi\left(\frac{1}{x}\right)}{x}dx + \frac{1}{s-1} - \frac{1}{s}.$$

Substituting $x = \frac{1}{u}$, $dx = -\frac{1}{u^2}$ this becomes

$$= \frac{1}{s(s-1)} + \int_1^\infty \frac{u^{-\frac{1}{2}s+\frac{1}{2}}\psi\left(u\right)}{u}du.$$

Substituting this into 2.3 we find

$$\zeta(s)\Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s} = \frac{1}{s(s-1)} + \int_1^\infty x^{\frac{1}{2}s-1}\psi(x) + x^{-\frac{1}{2}s-\frac{1}{2}}\psi\left(x\right)dx$$

as desired. □

3 Infinitely Many Zeros on the Critical Line

In this section we show a proof of Hardy's theorem that there are infinitely many zeros on the critical line.

The following Lemma relates an integral of the function $\Xi(t)$ to $\psi(e^{-2x})$. This identity will be at the center of the proof of Hardy's result.

Lemma 5. *We have that*

$$\int_0^\infty \left(t^2+\frac{1}{4}\right)^{-1}\Xi(t)\cos(xt)dt = \frac{1}{2}\pi\left(e^{\frac{1}{2}x} - 2e^{-\frac{1}{2}x}\psi\left(e^{-2x}\right)\right). \tag{3.1}$$

Proof. Let

$$Q(x) = \int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t)\cos(xt)dt.$$

Then since $\left(t^2 + \frac{1}{4}\right)^{-1}\Xi(t)\cos(xt)$ is an even function of t we see that

$$Q(x) = \frac{1}{2}\int_{-\infty}^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t)\cos(xt)dt.$$

Now, as $\left(t^2 + \frac{1}{4}\right)^{-1}\Xi(t)\sin(xt)$ is an odd function of t, its integral over the real line is zero, and hence

$$Q(x) = \frac{1}{2}\int_{-\infty}^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t)e^{ixt}dt.$$

Let $s = \frac{1}{2} + it$. Then

$$Q(x) = \frac{e^{-\frac{1}{2}xt}}{2i}\int_{\frac{1}{2}-\infty}^{\frac{1}{2}+\infty} \frac{1}{s(1-s)}\Xi(s)e^{xs}ds.$$

By 1.1, the definition of $\xi(s)$, we have

$$Q(x) = -\frac{e^{-\frac{1}{2}xt}}{4i}\int_{\frac{1}{2}-\infty}^{\frac{1}{2}+\infty} \zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}e^{xs}ds.$$

The function $\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}e^{xs}$ is meromorphic on $\mathbb{C}$ with poles at $s = 0, s = 1$. Hence if we move the line integral to the right of the line $\sigma = 1$, the change will be accounted for by substracting the residue at $s = 1$. That is, for $\sigma > 1$ we have

$$Q(x) = -\frac{e^{-\frac{1}{2}x}}{4i}\int_{\sigma-\infty}^{\sigma+\infty} \zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}e^{xs}ds + \frac{e^{-\frac{1}{2}x}}{4i}\cdot 2\pi i\mathrm{Res}\left(\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}e^{xs},\ 1\right).$$

The residue at $s = 1$ is computed to be e^x since $\zeta(s)$ has a simple pole with residue 1. Thus

$$Q(x) = \frac{e^{-\frac{1}{2}x}}{4i}\int_{\sigma-\infty}^{\sigma+\infty} \zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}e^{xs}ds + \frac{\pi}{2}e^{\frac{1}{2}x}.$$

By applying 2.1 with $y = e^{-2x}$ we see that

$$Q(x) = -\pi e^{-\frac{1}{2}x}\psi(e^{-2x}) + \frac{\pi}{2}e^{\frac{1}{2}x}$$

as desired. $\square$

Lemma 6. *For a every integer n we have*

$$\lim_{\alpha\to\frac{1}{4}\pi^+} \frac{d^{2n}}{d\alpha^{2n}}\left[e^{\frac{1}{2}i\alpha}\left(\frac{1}{2} + \psi\left(e^{2i\alpha}\right)\right)\right] = 0.$$

Proof. First, notice that

$$\psi(i+\delta) = \sum_{n=1}^{\infty} e^{-n^2\pi(i+\delta)} = \sum_{n=1}^{\infty} (-1)^n e^{-n^2\pi\delta}$$

and hence

$$\psi(i+\delta) = 2\psi(4\delta) - \psi(\delta). \tag{3.2}$$

As

$$\psi(x) = x^{-\frac{1}{2}}\psi\left(\frac{1}{x}\right) + \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}$$

by 2.2, we see that 3.2 becomes

$$\psi(i+\delta) = \frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{4\delta}\right) - \frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{\delta}\right) - \frac{1}{2}.$$

By expanding the series definition for $\psi(x)$ it follows that $\frac{1}{2} + \psi(i+\delta)$ and all of its derivatives tend to zero as $\delta \to 0$ with $\delta \in \mathbb{R}^+$. Hence they also go to zero along any route with angle $|\arg(\delta)| < \frac{1}{2}\pi$ since for any δ with $\Re(\delta) > 0$ we have that

$$\left|\sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{\delta}}\right| \leq \sum_{n=1}^{\infty} e^{-\pi n^2 \frac{\Re(\delta)}{|\delta|^2}} \leq \sum_{n=1}^{\infty} e^{-\pi n^2 \frac{1}{|\delta|}}.$$

Now, as $\alpha \to \frac{\pi}{4}^+$ implies that $e^{2i\alpha} \to i$ along any route with $|\arg(e^{2i\alpha} - i)| < \frac{1}{2}\pi$, the lemma is proven. □

Theorem 7. *$\Xi(t)$ has infinitely many zeros.*

Proof. Substituting $x = -i\alpha$ in 3.1 we find

$$\int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t)\cosh(\alpha t)dt = \frac{\pi}{2}\left(e^{-\frac{1}{2}i\alpha} - 2e^{\frac{1}{2}i\alpha}\psi\left(e^{2i\alpha}\right)\right).$$

$$= \pi\cos\frac{\alpha}{2} - \pi e^{\frac{1}{2}i\alpha}\left(\frac{1}{2} + \psi\left(e^{2i\alpha}\right)\right).$$

In 1908 Lindelof proved that $\zeta\left(\frac{1}{2} + it\right) + O\left(t^{\frac{1}{4}}\right)$ [3]. By Stirlings formula, $\Gamma\left(\frac{1}{4} + \frac{it}{2}\right) = O\left(e^{-\frac{1}{4}\pi t}\right)$, so that

$$\left(t^2 + \frac{1}{4}\right)^{-1} t^{2n}\Xi(t)\cosh(\alpha t) = O\left(t^{\frac{1}{4}+2n}e^{-\frac{1}{4}\pi + \alpha}\right).$$

Consequently, we can take the derivative with respect to α and move this underneath the integration sign provided $\alpha < \frac{1}{4}\pi$. Taking the derivative $2n$ times we see

$$\int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} t^{2n}\Xi(t)\cosh(\alpha t)dt = \frac{\pi(-1)^n}{2^{2n}}\cos\frac{\alpha}{2} - \pi\frac{d^{2n}}{d\alpha^{2n}}\left[e^{\frac{1}{2}i\alpha}\left(\frac{1}{2} + \psi\left(e^{2i\alpha}\right)\right)\right].$$

Taking the limit as $\alpha \to \frac{1}{4}\pi^+$ and applying 6 yields

$$\lim_{\alpha \to \frac{1}{4}\pi^+} \int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} t^{2n}\Xi(t)\cosh(\alpha t)dt = \frac{\pi(-1)^n}{2^{2n}} \cos\frac{\pi}{8}. \tag{3.3}$$

□

Suppose to get a contradiction that $\Xi(t)$ had only finitely many zero, and hence never changes sign for $t > T$ for some large T. Assume without loss of generality that $\Xi(t) > 0$. (The other case is handled identically) Let L be defined by

$$\lim_{\alpha \to \frac{1}{4}\pi^+} \int_T^\infty \left(t^2 + \frac{1}{4}\right)^{-1} t^{2n}\Xi(t)\cosh(\alpha t)dt = L.$$

Then since cosh is monotonically increasing on $[0, \infty)$, $T^{'} > T$ implies

$$\int_T^{T^{'}} \left(t^2 + \frac{1}{4}\right)^{-1} t^{2n}\Xi(t)\cosh(\alpha t)dt \leq L$$

where we can truncate the integral since the integrand is non-negative on $[T, \infty)$. As this holds for every $T^{'} > T$ and for every $\alpha \in [0, \frac{\pi}{4})$ we see that

$$\int_T^{T^{'}} \left(t^2 + \frac{1}{4}\right)^{-1} t^{2n}\Xi(t)\cosh\left(\frac{1}{4}\pi t\right) dt \leq L$$

and hence the integral

$$\int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} t^{2n}\Xi(t)\cosh\left(\frac{1}{4}\pi t\right) dt$$

is absolutely convergent. As cosh is monotonic, $\left(t^2 + \frac{1}{4}\right)^{-1} t^{2n}\Xi(t)\cosh\left(\frac{1}{4}\pi t\right)$ dominates $\left(t^2 + \frac{1}{4}\right)^{-1} t^{2n}\Xi(t)\cosh\left(\alpha t\right)$ for each $\alpha \in [0, \frac{\pi}{4})$ so that the dominated convergence theorem allows us to switch the order of the limit and the integral. Hence by 3.3 we have that for every n

$$\int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} t^{2n}\Xi(t)\cosh\left(\frac{1}{4}\pi t\right) dt = \frac{\pi(-1)^n}{2^{2n}} \cos\frac{\pi}{8}.$$

However this is impossible since the right hand side switches sign infinitely often. Let n be odd. Then the right hand side is strictly less than zero so that

$$\int_T^\infty \left(t^2 + \frac{1}{4}\right)^{-1} t^{2n}\Xi(t)\cosh\left(\frac{1}{4}\pi t\right) dt < -\int_0^T \left(t^2 + \frac{1}{4}\right)^{-1} t^{2n}\Xi(t)\cosh\left(\frac{1}{4}\pi t\right) dt.$$

Since T is fixed, we have that

$$\left| \int_0^T \left(t^2 + \frac{1}{4}\right)^{-1} t^{2n} \Xi(t) \cosh\left(\frac{1}{4}\pi t\right) dt \right| \leq T^{2n} \int_0^T \left(t^2 + \frac{1}{4}\right)^{-1} |\Xi(t)| \cosh \frac{1}{4}\pi t dt$$

and setting $R = \int_0^T \left(t^2 + \frac{1}{4}\right)^{-1} |\Xi(t)| \cosh \frac{1}{4}\pi t dt$ we see that

$$-\int_0^T \left(t^2 + \frac{1}{4}\right)^{-1} t^{2n} \Xi(t) \cosh\left(\frac{1}{4}\pi t\right) dt < RT^{2n}$$

where R is independant of n. Now, by assumption there exists $\epsilon > 0$ such that $\Xi(t)\left(t^2 + \frac{1}{4}\right)^{-1} > \epsilon$ for all $2T < t < 2T + 1$ so that

$$\int_T^\infty \left(t^2 + \frac{1}{4}\right)^{-1} t^{2n} \Xi(t) \cosh\left(\frac{1}{4}\pi t\right) dt \geq \int_{2T}^{2T+1} t^{2n} \epsilon dt \geq \epsilon (2T)^{2n}.$$

Thus

$$\epsilon (2T)^{2n} < RT^{2n}$$

for all n. However, this is equivalent to

$$2^{2n} < \frac{R}{\epsilon}$$

holding for all n, which is impossible since n can be taken arbitrarily large. Thus we have our contradiction, and the theorem is proven.

Part II

A Positive Proportion of the Zeros Lie on the Critical Line

In this part we show Selbergs proof that a positive proportion of the zeros of ζ lie on the critical line.

4 Outline of the proof

For each T, the goal is to put a lower bound on the number of zeros of $\Xi(t)$ with $t \leq T$. Rather than count the zeros of $\Xi(t)$ themselves, we will choose a small constant h, and put a lower bound on the number of intervals of the form $(nh, (n+1)h) \subset (0, T)$ which contain a zero. With this in mind, it then makes sense to look at

$$E^{'} = \left\{ 0 \leq t \leq T : \ \exists t^{'} \in (t, t+h) \text{ with } \Xi(t^{'}) = 0 \right\},$$

and attempt to find $m(E^{'})$, the size of $E^{'}$. This set however is not desirable, as the method we use here to detect zeros of $\Xi(t)$ is by examining sign changes. A sign change of the function $\Xi(t)$ on the interval $(t, t+h)$ implies there must be a zero, however the converse is not neccesarily true. Hence consider $E \subset E^{'}$ defined by

$$E = \{0 \le t \le T : \; \Xi(t) \text{ changes sign on } (t, t+h)\} .$$

The goal then becomes finding a suitable lower bound on $m(E)$. In particular, we will show that when $h = \frac{c}{\log T}$, $c > 0$, we must have $m(E) > BT$, $B > 0$. Once we prove this, Selbergs result that $N_0(T) > AT \log T$ follows. To see why, notice that of the intervals

$$(0, h), \; (h, 2h), \; (2h, 3h) \; \dots$$

at least

$$\frac{BT}{h} = BcT \log T$$

must contain a point of E. Since $t \in (nh, (n+1)h)$ and $t \in E$ implies that there is a zero in $(nh, (n+2)h)$, we see that

$$N_0(T) > \frac{1}{2} BcT \log T$$

where the factor of $\frac{1}{2}$ comes from the fact that each zero could be counted by two different intervals.

Proving this lower bound for $m(E)$ consists of multiple steps. First, notice that

$$E = \left\{ 0 \le t \le T : \left| \int_t^{t+h} \Xi(u) du \right| < \int_t^{t+h} |\Xi(u)| du \right\} .$$

As the function $\Xi(t)$ itself can be difficult to deal with, we look instead at $F(t) = \Xi(t)W(t)$ for some suitable function $W(t) > 0$. In particular the function $W(t)$ will be chosen so that $\Xi(t)W(t)$ is the fourier transform of some $f(y)$ which we can work with more easily. Since the zeros of $F(t)$ will correspond to zeros of $\Xi(t)$ we see that

$$E = \left\{ 0 \le t \le T : \left| \int_t^{t+h} F(u) du \right| < \int_t^{t+h} |F(u)| du \right\} .$$

The rest of the proof is then centered around finding bounds for integrals involving the functions $\int_t^{t+h} F(u) du$ and $\int_t^{t+h} |F(u)| du$. Specifically, we will find a way to bound the integral

$$\int_E \int_t^{t+h} F(u) du dt$$

from above and below, where the upper bound will introduce $m(E)$ by application of Cauchy-Schwarz. It is then from these upper and lower bounds that we are able to deduce $m(E) > BT$ when $h = \frac{c}{\log T}$.

The proof itself is divided into four major sections In the first section, $W(t)$ will be specified, along with $F(t)$ and its Fourier transform $f(y)$. In the second section the function $J(x,\theta)$ is introduced, which is related to $F(t)$. The purpose of this entire section becomes bounding $J(x,\theta)$ from above. This is by far the longest, and is the most technically difficult section, as many of the sums run over as many as 7 variables. The third section will be a series of corollaries to the preceeding upper bounds on $J(x,\theta)$, and in particular we will place bounds on

$$\int_{-\infty}^{\infty} |F(t)|^2 dt$$

and

$$\int_{-\infty}^{\infty} \left| \int_{t}^{t+h} F(u)du \right|^2 dt.$$

Some important lower bounds for the integrals of $F(t)$ and $|F(t)|$ are also derived. In the fourth section, we will prove the main result using the upper and lower bounds from the third section.

5 Preliminaries, and the function $W(t)$

Recall 2.1 which tells us that

$$\psi(y) = \frac{1}{4\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \zeta(s)\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} y^{-\frac{s}{2}} ds.$$

We are going to modify the integrand by multiplying by $\phi(s)\phi(1-s)$ for a suitable function ϕ. The reason we multiply by $\phi(s)\phi(1-s)$ rather that just $\phi(s)$ is to show explicitely that the symmetry around the line $\Re(s) = \frac{1}{2}$ will be preserved.

Define α_ν by

$$\frac{1}{\sqrt{\zeta(s)}} = \sum_{\nu=1}^{\infty} \alpha_\nu \nu^{-s}$$

where $\sigma > 1$ and $\alpha_1 = 1$. Notice that from the Euler product we have $\alpha_\mu \alpha_\nu = \alpha_{\mu\nu}$ if $(\nu,\mu) = 1$. Similarly define $\alpha_\nu^{'}$ by

$$\sqrt{\zeta(s)} = \sum_{\nu=1}^{\infty} \frac{\alpha_\nu^{'}}{\nu^s}$$

where $\sigma > 1$ and $\alpha_1^{'} = 1$. By expanding into Euler products, the fact that the series $(1-z)^{-\frac{1}{2}}$ termwise dominates the series for $(1-z)^{\frac{1}{2}}$ implies

$$|\alpha_\nu| \leq \alpha_\nu^{'} \leq 1. \tag{5.1}$$

Fix X and let

$$\beta_\nu = \left\{ \begin{array}{cc} \alpha_\nu \left(1 - \frac{\log \nu}{\log X}\right) & \text{if } \nu < X \\ 0 & \nu \geq X \end{array} \right\} \tag{5.2}$$

when $\nu < X$, and $\beta_\nu = 0$ if $\nu \geq X$. Notice

$$|\beta_\nu| \leq 1$$

for all ν. Then let

$$\phi(s) = \sum_{\nu=1}^{\infty} \beta_\nu \nu^{-s}.$$

With 2.1 in mind, consider the function

$$\Phi(z) = \frac{1}{4\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma\left(\frac{1}{2}s\right) \pi^{-\frac{1}{2}s} \zeta(s)\phi(s)\phi(1-s) z^s ds$$

where $\sigma > 1$. Moving the line of integration to $\sigma = \frac{1}{2}$, we see that

$$\Phi(z) = \frac{1}{2} z\phi(1)\phi(0) + \frac{1}{4\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \Gamma\left(\frac{1}{2}s\right) \pi^{-\frac{1}{2}s} \zeta(s)\phi(s)\phi(1-s) z^s ds$$

$$= \frac{1}{2} z\phi(1)\phi(0) - \frac{z^{\frac{1}{2}}}{2\pi} \int_{-\infty}^{\infty} \Xi(t) \left(t^2 + \frac{1}{4}\right)^{-1} |\phi\left(\frac{1}{2} + it\right)|^2 z^{it} dt.$$

On the other hand,

$$\Phi(z) = \frac{1}{4\pi i} \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \beta_\nu \beta_\mu \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma\left(\frac{1}{2}s\right) \pi^{-\frac{1}{2}s} \zeta(s) \frac{z^s}{\mu^s \nu^{1-s}} ds$$

which becomes

$$= \sum_{n=1}^{\infty} \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \frac{\beta_\nu \beta_\mu}{\nu} \exp\left(-\frac{\pi n^2 u^2}{z^2 \nu^2}\right)$$

by 2.1. Setting

$$z = e^{-i\left(\frac{1}{4}\pi - \frac{1}{2}\delta\right) - y}$$

it follows that the functions

$$F(t) = \frac{1}{\sqrt{2\pi}} \Xi(t) \left(t^2 + \frac{1}{4}\right)^{-1} |\phi\left(\frac{1}{2} + it\right)|^2 e^{\left(\frac{1}{4}\pi - \frac{1}{2}\delta\right)t} dt$$

and

$$f(y) = \frac{1}{2} z^{\frac{1}{2}} \phi(1)\phi(0) - z^{-\frac{1}{2}} \sum_{n=1}^{\infty} \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \frac{\beta_\nu \beta_\mu}{\nu} \exp\left(-\frac{\pi n^2 u^2}{z^2 \nu^2}\right)$$

are Fourier transforms.

This function $F(t)$ will be at the center of the rest of the proof, and refering to the outline, we are making the choice

$$W(t) = \frac{1}{\sqrt{2\pi}} \left(t^2 + \frac{1}{4}\right)^{-1} |\phi\left(\frac{1}{2} + it\right)|^2 z^{it} dt.$$

6 The functions $g(x)$ and $J(x,\theta)$

The purpose of this section is to define $g(x)$ and $J(x,\theta)$ and then find upper bounds for these two functions. In the next section, we will use the upper bound for $J(x,\theta)$ to bound several integrals of $F(t)$. We start with a lemma regarding Fourier transforms integrated over an interval of length h.

Lemma 8. *Suppose $F(u), f(y)$ are functions related by the Fourier formulas*

$$F(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{iyu} dy$$

$$f(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(u) e^{-iyu}.$$

If $f(y)$ is even and $F(u)$ is real we have that

$$\int_{-\infty}^{\infty} \left| \int_{t}^{t+h} F(u) du \right|^2 \leq 2h^2 \int_{0}^{\frac{1}{h}} |f(y)|^2 dy + 8 \int_{\frac{1}{h}}^{\infty} \frac{|f(y)|^2}{y^2} dy. \tag{6.1}$$

Proof. Integrating over $(t, t+h)$ and switching the order we obtain

$$\int_{t}^{t+h} F(u) du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) \int_{t}^{t+h} e^{iyu} du dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) \frac{e^{iyh} - 1}{iy} e^{iyt} dy$$

so that the functions

$$\int_{t}^{t+h} F(u) du$$

and

$$f(y) \frac{e^{iyh} - 1}{iy}$$

are Fourier transforms. By applying Parseval's formula we see that

$$\int_{-\infty}^{\infty} \left| \int_{t}^{t+h} F(u) du \right|^2 = -\infty |f(y)|^2 \frac{|e^{iyh} - 1|}{y^2} dy.$$

Notice that

$$|e^{iyh} - 1| = \sqrt{(\cos(yh) - 1)^2 + \sin^2(yh)}$$

$$= \sqrt{4 \left(\frac{1 - \cos(yh)}{2} \right)} = 2 \sin\left(\frac{yh}{2} \right)$$

by the half angle formula, so we have

$$\int_{-\infty}^{\infty}\left|\int_{t}^{t+h}F(u)du\right|^2 = 8\int_0^{\infty}|f(y)|^2\frac{\sin^2\left(\frac{yh}{2}\right)}{y^2}dy.$$

Splitting the integral on the right hand side into two parts, and using the bounds $|\sin(x)| \leq x$ and $|\sin(x)| \leq 1$ yields

$$\int_{-\infty}^{\infty}\left|\int_{t}^{t+h}F(u)du\right|^2 \leq 2h^2\int_0^{\frac{1}{h}}|f(y)|^2dy + 8\int_{\frac{1}{h}}^{\infty}\frac{|f(y)|^2}{y^2}dy$$

as desired. $\square$

6.1 $g(x)$ and its relation to $\int F(t)dt$.

Definition 9. Let

$$g(x) = \sum_{n=1}^{\infty}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}\frac{\beta_\mu\beta_\nu}{\nu}\exp\left(\frac{-\pi n^2\mu^2}{\nu^2}e^{-i\left(\frac{1}{2}\pi-\delta\right)}x^2\right)$$

so that

$$f(y) = \frac{1}{2}z^{\frac{1}{2}}\phi(1)\phi(0) - z^{-\frac{1}{2}}g\left(e^y\right)$$

where as before,

$$z = e^{-i\left(\frac{1}{4}\pi-\frac{1}{2}\delta\right)-y}.$$

The following proposition gives motivation for considering and bounding $g(x)$, as it arises naturally when we attempt to bound

$$\int_{-\infty}^{\infty}\left(\int_t^{t+h}F(u)du\right)^2dt.$$

Proposition 10. *Suppose $h \leq 1$, and let $G = e^{\frac{1}{h}}$. Then we have that*

$$\int_{-\infty}^{\infty}\left(\int_t^{t+h}F(u)du\right)^2dt < \frac{h^2}{2}|\phi(1)\phi(0)|^2\left(1+\frac{1}{G}\right) + 2h^2\int_1^G|g(x)|^2 + 2\int_1^G\frac{|g(x)|^2}{\log^2 x}dx \tag{6.2}$$

Proof. By 6.1

$$\int_{-\infty}^{\infty}\left|\int_{t}^{t+h}F(u)du\right|^2 =\leq 2h^2\int_0^{\frac{1}{h}}|f(y)|^2dy + 8\int_{\frac{1}{h}}^{\infty}\frac{|f(y)|^2}{y^2}dy \tag{6.3}$$

Setting $y = \log x$ we have that $z = e^{-i\left(\frac{1}{4}\pi-\frac{1}{2}\delta\right)-y} = \frac{1}{y}e^{-i\left(\frac{1}{4}\pi-\frac{1}{2}\delta\right)}$, and in particular $|z| = \frac{1}{y}$. Then if we set $G = e^{\frac{1}{H}}$ the first integral on the right hand side of 6.3 becomes

$$\int_0^{\frac{1}{H}} |f(y)|^2 dy = \int_1^G \left| \frac{e^{-i\left(\frac{1}{4}\pi-\frac{1}{2}\delta\right)}}{2x}\phi(1)\phi(0) - g(x)\right|^2 dx.$$

Then we have that

$$\int_0^{\frac{1}{H}} |f(y)|^2 dy \leq 2\int_1^G \frac{|\phi(1)\phi(0)|^2}{4x^2}dx + 2\int_1^G |g(x)|^2 dx < \frac{1}{2}|\phi(1)\phi(0)|^2 + 2\int_1^G |g(x)|^2 dx.$$

We can bound the second integral in a similar manner to find

$$8\int_{\frac{1}{h}}^{\infty} \frac{|f(y)|^2}{y^2}dy < \frac{|\phi(1)\phi(0)|^2}{2G\log^2 G} + 2\int_G^{\infty} \frac{|g(x)|^2}{\log^2 x}dx.$$

As $\frac{1}{\log^2 G} = h^2$, we have the desired result. $\square$

6.2 The Function $J(x, \theta)$.

To be able to bound $g(x)$ and its integrals of the form $\int |g(x)|^2 dx$ as they appear in 10, we consider

$$J(x,\theta) = \int_x^{\infty} |g(u)|^2 u^{-\theta} du$$

where $0 < \theta \leq \frac{1}{2}$, and $x \geq 1$. The goal of this subsection is to show that if $X = \delta^{-c}$ with $0 < c \leq \frac{1}{8}$ then

$$J(x,\theta) = O\left(\frac{1}{\delta^{\frac{1}{2}}\theta x^{\theta}\log X}\right) \tag{6.4}$$

uniformely with respect to θ.

Notice that

$$J(x,\theta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k\lambda\mu\nu} \frac{\beta_k\beta_\lambda\beta_\nu\beta_\mu}{\nu\lambda}R$$

where

$$R = \int_x^{\infty} \exp\left\{-\pi\left(\frac{m^2k^2}{\lambda^2} + \frac{n^2\mu^2}{\nu^2}\right)u^2\sin\delta + i\pi\left(\frac{m^2k^2}{\lambda^2} - \frac{n^2\mu^2}{\nu^2}\right)u^2\cos\delta\right\}\frac{du}{u^{\theta}}$$

from the definition of $g(x)$. Let Σ_1 denote the sum of those terms in which

$$\frac{mk}{\lambda} = \frac{n\mu}{\nu}$$

and Σ_2 the remainder. That is

$$\Sigma_2 = J(x,\theta) - \Sigma_1.$$

To prove 6.4 we will bound Σ_2 and Σ_1 seperately.

6.2.1 Bounding Σ_1.

Here we prove that when $X = \delta^{-c}$ with $0 < c \leq \frac{1}{8}$

$$\Sigma_1 = O\left(\frac{1}{\delta^{\frac{1}{2}}\theta x^{\theta}\log X}\right).$$

For each quadruplet k, ν, λ, μ let $q = (k\nu, \lambda\mu)$ so that $k\nu = aq$ and $\lambda\mu = bq$ for some a, b with $(a, b) = 1$. When $\frac{mk}{\lambda} = \frac{n\mu}{\nu}$ we have that $ma = nb$ and then $n = ra$, $m = rb$. This allows us to rewrite the sum of n and m as a single sum over r, and hence

$$\Sigma_1 = \sum_{k\lambda\mu\nu} \frac{\beta_k\beta_\lambda\beta_\nu\beta_\mu}{\nu\lambda} \sum_{r=1}^{\infty} \int_x^{\infty} \exp\left\{-2\pi\left(\frac{r^2k^2\mu^2}{q^2}\right)u^2\sin\delta\right\}\frac{du}{u^{\theta}}. \tag{6.5}$$

Definition 11. Let

$$S(\theta) = \sum_{k\lambda\mu\nu}\left(\frac{q}{k\mu}\right)^{1-\theta}\frac{\beta_k\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu}$$

where $q = \gcd(k\nu, \lambda\mu)$.

Lemma 12. *We have that*

$$\Sigma_1 = \frac{S(0)}{2(2\sin\delta)^{\frac{1}{2}}\theta x^{\theta}} + \frac{Q_1(\theta)}{\theta}(2\pi\sin\delta)^{\frac{1}{2}\theta-\frac{1}{2}}S(\theta) + O\left(\frac{x^{1-\theta}\log\left(\frac{X}{\delta}\right)}{\theta}X^2\log^2 X\right) \tag{6.6}$$

where $Q_1(\theta)$ is some bounded function of θ.

Proof. First, we will rewrite the sum over r in 6.5. Notice that

$$\sum_{r=1}^{\infty}\int_x^{\infty} e^{-r^2u^2\eta}\frac{du}{u^{\theta}} = \eta^{\frac{1}{2}\theta-\frac{1}{2}}\sum_{r=1}^{\infty}\frac{1}{r^{1-\theta}}\int_{xr\sqrt{\eta}}^{\infty} e^{-y^2}\frac{dy}{y^{\theta}}$$

$$= \eta^{\frac{1}{2}\theta-\frac{1}{2}}\int_{x\sqrt{\eta}}^{\infty}\frac{e^{-y^2}}{y^{\theta}}\left(\sum_{r\leq y/(x\sqrt{\eta})}\frac{1}{r^{1-\theta}}\right)dy.$$

Since

$$\sum_{r\leq y/(x\sqrt{\eta})}\frac{1}{r^{1-\theta}} = \frac{1}{\theta}\left(\frac{y}{x\sqrt{\eta}}\right)^{\theta} - \frac{1}{\theta} + Q(\theta) + O\left(\left(\frac{y}{x\sqrt{\eta}}\right)^{\theta-1}\right)$$

where $Q(\theta)$ is a bounded function of θ, we obtain

$$\sum_{r=1}^{\infty}\int_x^{\infty} e^{-r^2u^2\eta}\frac{du}{u^{\theta}} = \frac{1}{\theta x^{\theta}\sqrt{\eta}}\left(\int_0^{\infty} e^{-y^2}dy + O(x\sqrt{\eta})\right) - \frac{\eta^{\frac{1}{2}\theta-\frac{1}{2}}}{\theta}\left(\int_0^{\infty} e^{-y^2}y^{-\theta}dy + O\left((x\sqrt{\eta})^{1-\theta}\right)\right)$$

$$+\eta^{\frac{1}{2}\theta-\frac{1}{2}}Q(\theta)\left(\int_0^{\infty} e^{-y^2}y^{-\theta}dy + O\left(x\sqrt{\eta}\right)^{1-\theta}\right) + O\left(x^{1-\theta}\log\left(2+\eta^{-1}\right)\right)$$

$$= \frac{\sqrt{\pi}}{2\theta x^\theta \eta^{\frac{1}{2}}} + \frac{Q_1(\theta)\eta^{\frac{1}{2}\theta-\frac{1}{2}}}{\theta} + O\left(\frac{x^{1-\theta}}{\theta}\log\left(2+\eta^{-1}\right)\right).$$

Setting

$$\eta = \frac{2\pi k^2\mu^2\sin\delta}{q^2}$$

it follows that

$$\Sigma_1 = \frac{S(0)}{2(2\sin\delta)^{\frac{1}{2}}\theta x^\theta} + \frac{Q_1(\theta)}{\theta}(2\pi\sin\delta)^{\frac{1}{2}\theta-\frac{1}{2}}S(\theta) + O\left(\frac{x^{1-\theta}}{\theta}\sum_{k\lambda\mu\nu}\frac{|\beta_\lambda\beta_\mu\beta_\nu\beta_k|}{\nu\lambda}\log\left(2+\eta^{-1}\right)\right).$$

Since every non-zero term has each of $\lambda, k, \mu, \nu \leq X$, we see that

$$\log(2+\eta^{-1}) = \log\left(2+\frac{q^2}{2\pi k^2\mu^2\sin\delta}\right) = O\left(\log\frac{X}{\delta}\right)$$

and hence

$$\sum_{k\lambda\mu\nu}\frac{|\beta_\lambda\beta_\mu\beta_\nu\beta_k|}{\nu\lambda}\log\left(2+\eta^{-1}\right) = O\left(\log\frac{X}{\delta}X^2\log^2 X\right)$$

as desired $\square$

Given that we can write Σ_1 as in 6.6, it is sufficient to find a suitable upper bound of $S(\theta)$.

Define $\phi_a(n)$ by

$$\sum_{n=1}^{\infty}\frac{\phi_a(n)}{n^s} = \frac{\zeta(s-a-1)}{\zeta(s)}$$

so that

$$\phi_a(n) = n^{1+a}\sum_{m|n}\frac{\mu(m)}{m^{1+a}} = n^{1+a}\prod_{p|n}\left(1-\frac{1}{p^{1+a}}\right). \tag{6.7}$$

Then

$$q^{1-\theta} = \sum_{\rho|q}\phi_{-\theta}(\rho) = \sum_{\rho|k\nu,\ \rho|\lambda\mu}\phi_{-\theta}(\rho).$$

Consequently,

$$S(\theta) = \sum_{k\nu\mu\lambda}\frac{1}{k^{1-\theta}\mu^{1-\theta}} \sum_{\substack{k,\nu,\lambda,\mu \\ \rho|k\nu,\ \rho|\lambda\mu}}\phi_{-\theta}(\rho)\frac{\beta_k\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu}$$

and by rearranging the order of summation we have

$$S(\theta) = \sum_{\rho<X^2}\phi_{-\theta}(\rho)\left(\sum_{\substack{k,\nu \\ \rho|k\nu}}\frac{\beta_k\beta_\nu}{k^{1-\theta}\nu}\right)^2. \tag{6.8}$$

For each k, ν let d, d_1 be divisors of ρ that satisfy be $k = dk^{'}$, $\nu = d_1 v^{'}$ where $\gcd(k^{'}, \rho) = 1$ and $(\nu^{'}, \rho) = 1$. Then

$$\sum_{\substack{k,\nu \\ \rho | k\nu}} \frac{\beta_k \beta_\nu}{k^{1-\theta}\nu} = \sum_{\substack{d, d_1 \\ \rho | d, d_1}} \frac{1}{d^{1-\theta} d_1} \sum_{k'} \frac{\beta_{dk'}}{(k')^{1-\theta}} \sum_{\nu'} \frac{\beta_{d_1\nu'}}{\nu'}.$$

Now, by 5.2, when $(k', \rho) = 1$ we have that

$$\beta_{dk'} = \frac{\alpha_d \alpha_{k'}}{\log X} \log \frac{X}{dk'}$$

so that

$$\sum_{\substack{k,\nu \\ \rho | k\nu}} \frac{\beta_k \beta_\nu}{k^{1-\theta}\nu} = \frac{1}{\log^2 X} \sum_{\substack{d, d_1 \\ \rho | d, d_1}} \frac{\alpha_d \alpha_{d_1}}{d^{1-\theta} d_1} \sum_{k' \le \frac{X}{d}} \frac{\alpha_{k'}}{(k')^{1-\theta}} \log \frac{X}{dk'} \sum_{\nu' \le \frac{X}{d_1}} \frac{\alpha_{\nu'}}{\nu'} \log \frac{X}{d_1 \nu'}. \tag{6.9}$$

The next three lemma focus on bounding the right hand side of 6.9. By doing so, and combining this upper bound with 6.8 we will find an upper bound for $S(\theta)$, and hence by 6.6 for Σ_1 as well.

Lemma 13. *We have*

$$\sum_{k \le X/d} \frac{\alpha_k}{k^{1-\theta}} \log \frac{X}{kd} = O\left(\left(\frac{X}{d}\right)^{\theta} \log^{\frac{1}{2}} \frac{X}{d} \prod_{p|\rho} \left(1 + \frac{1}{p}\right)^{\frac{1}{2}} \right). \tag{6.10}$$

uniformly with respect to θ.

Proof. As, the only pole of

$$\frac{x^s}{s^2} = \frac{e^{s \log x}}{s^2}$$

is at $s = 0$ with residue $\log x$, it follows from the residue theorem that

$$\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{x^s}{s^2} ds = \left\{ \begin{array}{cc} 0 & 0 < x \le 1 \\ \log x & x > 1 \end{array} \right\}. \tag{6.11}$$

The two different possibilities arise since we close the contour in a direction dependant on the sign of $\log x$. Now, as

$$\sum_{\substack{k' \\ (k', \rho) = 1}} \frac{\alpha_{k'}}{(k')^{1-\theta+s}} = \prod_{\substack{p \\ (p, \rho) = 1}} \left(1 - \frac{1}{p^{1-\theta+s}}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{\zeta(1-\theta+s)}} \prod_{p|\rho} \left(1 - \frac{1}{p^{1-\theta}}\right)$$

we can apply 6.11 to find

$$\sum_{k \le X/d} \frac{\alpha_k}{k^{1-\theta}} \log \frac{X}{kd} = \sum_{k \le X/d} \frac{\alpha_k}{k^{1-\theta}} \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{1}{s^2} \left(\frac{X}{kd}\right)^s ds,$$

and upon switching the order of summation and integration this becomes

$$\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{1}{s^2} \left(\frac{X}{d}\right)^s \frac{1}{\sqrt{\zeta(1-\theta+s)}} \prod_{p|\rho} \left(1 - \frac{1}{p^{1-\theta}}\right) \frac{x^s}{s^2} ds. \tag{6.12}$$

The integrand has singularities at $s = 0$ and $s = \theta$. Now, lets split into cases based on the size of θ.

If $\theta \ge \left(\log\left(\frac{X}{d}\right)\right)^{-1}$, we can move the line of integration to the line $\Re(s) = \theta$, with a small semicircle tending to zero at $s = \theta$. Notice we have that

$$\left|\frac{1}{\zeta(1+it)}\right| < A|t|$$

for all t, as well as

$$\prod_{p|\rho} \left(1 - \frac{1}{p^{1-\theta+s}}\right)^{-1} = O\left(\prod_{p|\rho} \left(1 + \frac{1}{p^{1-\theta+s}}\right)\right) = O\left(\prod_{p|\rho} \left(1 + \frac{1}{p}\right)\right). \tag{6.13}$$

Consequently 6.12 is

$$O\left(\left(\frac{X}{d}\right)^\theta \prod_{p|\rho} \left(1 + \frac{1}{p}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{|t|^{\frac{1}{2}}}{\theta^2 + t^2} dt\right) = O\left(\left(\frac{X}{d}\right)^\theta \prod_{p|\rho} \left(1 + \frac{1}{p}\right)^{\frac{1}{2}} \frac{1}{\theta^{\frac{1}{2}}}\right)$$

and the stated result follows.

If $\theta < \left(\log\left(\frac{X}{d}\right)\right)^{-1}$, we take the same line integral as before, modified by going around the right hand side of the circle $|s| = 2\left(\log\left(\frac{X}{d}\right)\right)^{-1}$. On this circle,

$$\left|\left(\frac{X}{d}\right)^s\right| \le e^2,$$

and 6.13 holds as before. As

$$|\zeta(1-\theta+s)| > A \log\left(\frac{X}{d}\right)$$

we see that the integral around the circle is

$$O\left(\log^{-\frac{1}{2}} \frac{X}{d} \prod_{p|\rho} \left(1 + \frac{1}{p}\right)^{-\frac{1}{2}} \int \left|\frac{ds}{s^2}\right|\right) = O\left(\log^{\frac{1}{2}} \frac{X}{d} \prod_{p|\rho} \left(1 + \frac{1}{p}\right)^{\frac{1}{2}}\right).$$

The integral along the part of the line $\sigma = \theta$ above the circle is

$$O\left(\left(\frac{X}{d}\right)^{\theta}\prod_{p|\rho}\left(1+\frac{1}{p}\right)^{\frac{1}{2}}\int_{A(\log(X/d))^{-1}}\frac{dt}{t^{\frac{3}{2}}}\right) = O\left(\left(\frac{X}{d}\right)^{\theta}\log^{\frac{1}{2}}\frac{X}{d}\prod_{p|\rho}\left(1+\frac{1}{p}\right)^{\frac{1}{2}}\right).$$

Thus the lemma is proven in all cases. □

Lemma 14.

$$\sum_{\substack{d,\,d_1\\ \rho|dd_1}}\frac{|\alpha_d\alpha_{d_1}|}{dd_1} = O\left(\frac{1}{\rho}\prod_{p|\rho}\left(\frac{1}{1+p}\right)\right). \tag{6.14}$$

Proof. Let $\alpha_d^{'}$ be defined as before so that

$$\sqrt{\zeta(s)} = \sum_{n=1}^{\infty}\alpha_n^{'}n^{-s}.$$

Then by 5.1 we have

$$\sum_{\substack{d,\,d_1\\ \rho|dd_1}}\frac{|\alpha_d\alpha_{d_1}|}{dd_1} \leq \sum_{\substack{d,\,d_1\\ \rho|dd_1}}\frac{\alpha_d^{'}\alpha_{d_1}^{'}}{dd_1} = \sum_{\substack{D\\ \rho|D}}\frac{1}{D}\sum_{d|D}\alpha_d^{'}\alpha_{D/d}^{'}.$$

As $\alpha_d^{'}$ are the coefficients of $\sqrt{\zeta(s)}$, $\sum_{d|D}\alpha_d^{'}\alpha_{D/d}^{'} = 1$ so that

$$\sum_{\substack{d,\,d_1\\ \rho|dd_1}}\frac{|\alpha_d\alpha_{d_1}|}{dd_1} \leq \sum_{\substack{D\\ \rho|D}}\frac{1}{D} = \frac{1}{\rho}\prod_{p|\rho}\left(1-\frac{1}{p}\right)^{-1} = O\left(\frac{1}{\rho}\prod_{p|\rho}\left(1+\frac{1}{p}\right)\right)$$

as desired. □

Lemma 15. *We have*

$$\sum_{\substack{k,\ \nu\\ \rho|k\nu}}\frac{\beta_k\beta_\nu}{k^{1-\theta}\nu} = O\left(\frac{X^{\theta}}{\rho\log X}\prod_{p|\rho}\left(1+\frac{1}{p}\right)^{2}\right). \tag{6.15}$$

Proof. By 6.10 we see that

$$\sum_{\nu'}\frac{\alpha_{\nu'}}{\nu'}\log\frac{X}{d_1\nu'} = O\left(\log^{\frac{1}{2}}\frac{X}{d_1}\prod_{p|\rho}\left(1+\frac{1}{p}\right)^{\frac{1}{2}}\right)$$

and

$$\sum_{k'\le\frac{X}{d}}\frac{\alpha_{k'}}{(k')^{1-\theta}}=O\left(\left(\frac{X}{d}\right)^{\theta}\log^{\frac{1}{2}}\frac{X}{d}\prod_{p|\rho}\left(1+\frac{1}{p}\right)^{\frac{1}{2}}\right)$$

so that 6.9 becomes

$$\sum_{\substack{k,\ \nu\\ \rho|k\nu}}\frac{\beta_k\beta_\nu}{k^{1-\theta}\nu}=O\left(\frac{1}{\log^2 X}\sum_{\substack{d,d_1\\ \rho|d,d_1}}\frac{|\alpha_d\alpha_{d_1}|}{d^{1-\theta}d_1}\left(\frac{X}{d}\right)^{\theta}\log^{\frac{1}{2}}\frac{X}{d}\log^{\frac{1}{2}}\frac{X}{d_1}\prod_{p|\rho}\left(1+\frac{1}{p}\right)\right).$$

This equals

$$O\left(\frac{X^\theta}{\log X}\prod_{p|\rho}\left(1+\frac{1}{p}\right)\sum_{\substack{d,d_1\\ \rho|d,d_1}}\frac{|\alpha_d\alpha_{d_1}|}{d^1d_1}\right)$$

and by 6.14 we conclude

$$\sum_{\substack{k,\ \nu\\ \rho|k\nu}}\frac{\beta_k\beta_\nu}{k^{1-\theta}\nu}=O\left(\frac{X^\theta}{\rho\log X}\prod_{p|\rho}\left(1+\frac{1}{p}\right)^2\right)$$

as desired □

Lemma 16.

$$S(\theta)=O\left(\frac{X^{2\theta}}{\log X}\right)\tag{6.16}$$

uniformely with respect to θ. In particular

$$S(0)=O\left(\frac{1}{\log X}\right).$$

Proof. Combining 6.8 and 6.15 yields

$$S(\theta)=O\left(\frac{X^{2\theta}}{\log^2 X}\sum_{\rho<X^2}\frac{\phi_{-\theta}(\rho)}{\rho^2}\prod_{p|\rho}\left(1+\frac{1}{p}\right)^4\right).$$

By applying 6.7 we see that

$$S(\theta) = O\left(\frac{X^{2\theta}}{\log^2 X}\sum_{\rho<X^2}\frac{1}{\rho^{1+\theta}}\prod_{p|\rho}\left(1+\frac{1}{p}\right)^4\right).$$

Since

$$\prod_{p|\rho}\left(1+\frac{1}{p}\right)^4 = O\left(\prod_{p|\rho}\left(1+\frac{4}{p}\right)\right) = O\left(\prod_{p|\rho}\left(1+\frac{1}{p^{\frac{1}{2}}}\right)\right) = O\left(\sum_{n|\rho}\frac{1}{n^{\frac{1}{2}}}\right)$$

we have

$$S(\theta) = O\left(\frac{X^{2\theta}}{\log^2 X}\sum_{\rho<X^2}\frac{1}{\rho^{1+\theta}}\sum_{n|\rho}\frac{1}{n^{\frac{1}{2}}}\right).$$

Thus

$$\begin{aligned}S(\theta) &= O\left(\frac{X^{2\theta}}{\log^2 X}\sum_{n\le X^2}\sum_{\rho\le\frac{X^2}{n}}\frac{1}{(n\rho)^{1+\theta}n^{\frac{1}{2}}}\right)\\ &= O\left(\frac{X^{2\theta}}{\log^2 X}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}+\theta}}\sum_{\rho\le\frac{X^2}{n}}\frac{1}{\rho^{1+\theta}}\right)\\ &= O\left(\frac{X^{2\theta}}{\log^2 X}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}\sum_{\rho\le\frac{X^2}{n}}\frac{1}{\rho^{1}}\right)\\ &= O\left(\frac{X^{2\theta}}{\log X}\right).\end{aligned}$$

$\square$

In what follows, let $X = \delta^{-c}$, $h = (a\log X)^{-1}$ where a, c are suitable positive constants. Then $G = X^a = \delta^{-ac}$. If $x \le G$, the last two terms can be ommited in comparison with the first if $GX^2 = O(\delta^{-\frac{1}{4}})$, i.e. if $(a+2)c \le \frac{1}{4}$.

Lemma 17. *Estimation of* Σ_1*. When* $X = \delta^{-c}$*,* $0 < c \le \frac{1}{8}$*we have that*

$$\Sigma_1 = O\left(\frac{1}{\delta^{\frac{1}{2}}\theta x^\theta \log X}\right). \tag{6.17}$$

Proof. By 6.6 along with 6.16 we have that

$$\Sigma_1 = O\left(\frac{1}{\delta^{\frac{1}{2}}\theta x^\theta\log X}\right) + O\left(\frac{\left(\delta^{\frac{1}{2}}xX^2\right)^\theta}{\delta^{\frac{1}{2}}\theta x^\theta\log X}\right) + O\left(\frac{x^{1-\theta}\log\left(\frac{X}{\delta}\right)}{\theta}X^2\log^2 X\right).$$

Since $X = \delta^{-c}$ with $0 < c \leq \frac{1}{8}$ this bcomes

$$\Sigma_1 = O\left(\frac{1}{\delta^{\frac{1}{2}}\theta x^\theta \log X}\right).$$

$\square$

6.2.2 Bounding Σ_2.

Lemma 18. *If P and Q are positive, and $x \geq 1$ we have*

$$\int_x^\infty e^{-Pu^2+iQu^2}\frac{du}{u^\theta} = O\left(\frac{e^{-P}}{x^\theta Q}\right) \tag{6.18}$$

Proof. Since $\int_x^\infty e^{-Pu^2+iQu^2}\frac{du}{u^\theta} = \frac{1}{2}\int_{x^2}^\infty \frac{e^{-Pv}}{v^{\frac{1}{2}\theta+\frac{1}{2}}}e^{iQv}dv$ $\square$

Lemma 19. *When $X = \delta^{-c}$ with $0 < c \leq \frac{1}{8}$ we have that*

$$\Sigma_2 = O\left(\frac{X^4}{x^\theta}\log^2\frac{1}{\delta}\right). \tag{6.19}$$

Proof. Letting $P = \pi\left(\frac{m^2k^2}{\lambda^2} + \frac{n^2\mu^2}{\nu^2}\right)\sin\delta$ and $Q = \pi\left(\frac{m^2k^2}{\lambda^2} - \frac{n^2\mu^2}{\nu^2}\right)\cos\delta$ in 6.18 it follows from the definition of Σ_2 that

$$\Sigma_2 = O\left(\frac{1}{x^\theta}\sum_{k\lambda\mu\nu}\frac{|\beta_k\beta_\lambda\beta_\nu\beta_\mu|}{\lambda\nu}\sum_{mn}^{*}\left|\frac{m^2k^2}{\lambda^2} - \frac{n^2u^2}{\nu^2}\right|^{-1}\exp\left(-\pi\left(\frac{m^2k^2}{\lambda^2} + \frac{n^2\mu^2}{\nu^2}\right)\sin\delta\right)\right)$$

where

$$\sum_{mn}^{*}$$

denotes the fact that the sum does not range over all m, n. Notice that by symmetry, the cases $\frac{mk}{\lambda} > \frac{n\mu}{\nu}$ and $\frac{mk}{\lambda} < \frac{n\mu}{\nu}$ are identical, so that

$$\Sigma_2 = O\left(\frac{1}{x^\theta}\sum_{k\lambda\mu\nu}\frac{|\beta_k\beta_\lambda\beta_\nu\beta_\mu|}{\lambda\nu}\sum_{m=1}^{\infty}e^{-\pi m^2k^2\lambda^{-2}\sin\delta}\sum_{n<mk\nu/\lambda\mu}\left(\frac{m^2k^2}{\lambda^2} - \frac{n^2\mu^2}{\nu^2}\right)^{-1}\right).$$

The presence of the $|\beta_k\beta_\lambda\beta_\nu\beta_\mu|$ term means that each nonzero term has all of $k, \nu, \lambda, \mu \leq X$. Hence ignore all quadruplets with $k\nu/\lambda\mu \geq X^2$ since that implies that one of $k, \nu \geq X$. Then

$$\sum_{n<mk\nu/\lambda\mu}\frac{1}{mk\nu - n\lambda\mu} \leq 1 + \frac{1}{\lambda\mu} + \frac{1}{2\lambda\mu} + \cdots = 1 + O\left(\frac{\log mX}{\lambda\mu}\right)$$

and also

$$\frac{m^2k^2}{\lambda^2} - \frac{n^2\mu^2}{\nu^2} \geq \frac{mk}{\lambda}\left(\frac{mk}{\lambda} - \frac{n\mu}{\nu}\right) = \frac{mk\,(mk\nu - n\lambda\mu)}{\lambda^2\nu}.$$

Thus we have that

$$\sum_{m=1}^{\infty} e^{-\pi m^2k^2\lambda^{-2}\sin\delta} \sum_{n<mk\nu/\lambda\mu} \left(\frac{m^2k^2}{\lambda^2} - \frac{n^2\mu^2}{\nu^2}\right)^{-1}$$

$$= O\left(\frac{\lambda^2\nu}{k}\sum_{m=1}^{\infty}\left(\frac{1}{m} + \frac{\log(mX)}{m\lambda u}\right)e^{-\pi m^2k^2\lambda^{-2}\sin\delta}\right)$$

$$= O\left(\frac{\lambda^2\nu}{k}\left(1 + \frac{\log X}{\lambda\mu}\right)\log\frac{X^2}{\delta} + \frac{\lambda\nu}{k\mu}\log^2\frac{X^2}{\delta}\right)$$

$$= O\left(\frac{\lambda^2\nu}{k}\log\frac{1}{\delta} + \frac{\lambda\nu}{k\mu}\log^2\frac{1}{\delta}\right)$$

since $X = \delta^{-c}$ with $0 < c < \frac{1}{8}$. Hence

$$\Sigma_2 = O\left(\frac{1}{x^\theta}\sum_{k\lambda\mu\nu}\left(\frac{\lambda}{k}\log\frac{1}{\delta} + \frac{1}{k\mu}\log^2\frac{1}{\delta}\right)\right) = O\left(\frac{X^4}{x^\theta}\log^2\frac{1}{\delta}\right)$$

as desired. $\square$

Lemma 20. *The upper bound 6.4 holds. That is, we have*

$$J(x,\theta) = O\left(\frac{1}{\delta^{\frac{1}{2}}\theta x^\theta \log X}\right).$$

Proof. This follows from combining 6.17 and 6.19 along with the fact that $X = \delta^{-c}$ with $0 < c < \frac{1}{8}$. $\square$

7 Bounding $\int F(t)$ and $\int |F(t)|$.

Similar to the previous section, we assume that $X = \delta^{-c}$ for a positive constant c. Eventually we will choose $h = (a\log X)^{-1}$.

7.0.3 The integrals $\int_{-\infty}^{\infty}\left|\int_t^{t+h}F(u)du\right|^2 dt$ and $\int_{-\infty}^{\infty}|F(t)|^2dt$.

In this subsection we find upper bounds for $\int_{-\infty}^{\infty}|F(t)|^2dt$ and $\int_{-\infty}^{\infty}\left|\int_t^{t+h}F(u)du\right|^2 dt$ by using 6.2 and 6.4.

Lemma 21.

$$\int_{-\infty}^{\infty}\left|\int_t^{t+h} F(u)du\right|^2 dt = O\left(\frac{h}{\delta^{\frac{1}{2}}\log X}\right). \tag{7.1}$$

Proof.

$$J(x,\theta) = \int_x^{\infty} |g(u)|^2 u^{-\theta} du$$

Since

$$J(x,\theta) = O\left(\frac{1}{\delta^{\frac{1}{2}}\theta x^{\theta}\log X}\right)$$

it follows that

$$\int_1^G |g(x)|^2 dx = -\int_1^G x^{\theta}\frac{\partial T}{\partial x}dx = -x^{\theta}J\Big|_1^G + \theta\int_1^G x^{\theta-1}T dx$$

$$= O\left(\frac{1}{\delta^{\frac{1}{2}}\theta\log X}\right) + O\left(\theta\int_1^G \frac{dx}{\delta^{\frac{1}{2}}\theta x\log X}\right).$$

Also,

$$\int_0^{\frac{1}{2}} J(G,\theta)d\theta = \int_G^{\infty}|g(x)|^2dx\int_0^{\frac{1}{2}}\theta x^{-\theta}d\theta$$

$$= \int_G^{\infty}|g(x)|^2\left(\frac{1}{\log^2 x} - \frac{1}{2x^{\frac{1}{2}}\log x} - \frac{1}{x^{\frac{1}{2}}\log^2 x}\right)dx$$

$$\geq \int_G^{\infty}\frac{|g(x)|^2}{\log^2 x}dx - \frac{3}{2}\int_G^{\infty}\frac{|g(x)|^2}{x^{\frac{1}{2}}}dx$$

since $G = e^{\frac{1}{h}} \geq e$. (We have been assuming $h \leq 1$ throughout.) Hence

$$\int_G^{\infty}\frac{|g(x)|^2}{\log^2 x} \leq \int_0^{\frac{1}{2}}\theta J(G,\theta)d\theta + \frac{3}{2}J(G,\frac{1}{2})$$

$$= O\left(\int_0^{\frac{1}{2}}\frac{d\theta}{\delta^{\frac{1}{2}}G^{\theta}\log X}\right) + O\left(\frac{1}{\delta^{\frac{1}{2}}G^{\frac{1}{2}}\log X}\right) = O\left(\frac{1}{\delta^{\frac{1}{2}}\log G\log X}\right).$$

By 6.2 we have that

$$\int_{-\infty}^{\infty}\left(\int_t^{t+h}F(u)du\right)^2 dt < \frac{1}{2}|\phi(1)\phi(0)|^2\left(1+\frac{1}{G\log^2 G}\right)+2\int_1^G|g(x)|^2+2\int_1^G\frac{|g(x)|^2}{\log^2 x}dx$$

Since $\phi(0) = O(x)$ and $\phi(1) = O(\log X)$, we then have that

$$\int_{-\infty}^{\infty}\left|\int_t^{t+h}F(u)du\right|^2 dt = O\left(\frac{h}{\delta^{\frac{1}{2}}\log X}\right)$$

since $X = \delta^{-c}$ with $0 < c \leq \frac{1}{8}$. □

Lemma 22.

$$\int_{-\infty}^{\infty} |F(t)|^2 dt = O\left(\frac{\log(1/\delta)}{\delta^{\frac{1}{2}} \log X}\right) \tag{7.2}$$

Proof. By Plancherels formula, the left hand side becomes

$$2\int_0^{\infty} |f(y)|^2 dy = 2\int_1^{\infty} \left| \frac{e^{-i\left(\frac{1}{4}\pi - \frac{1}{2}\delta\right)}}{2x} \phi(1)\phi(0) - g(x) \right|^2 dx$$

$$\leq 4\int_1^{\infty} |g(x)|^2 dx + O(X^2 \log^2 X).$$

Taking $x = 1$, $\theta = \frac{1}{\log(1/\delta)}$ in

$$J(x, \theta) = O\left(\frac{1}{\delta^{\frac{1}{2}} \theta x^{\theta} \log X}\right)$$

yields

$$\int_1^{\infty} |g(u)|^2 e^{\log u / \log \delta} du = O\left(\frac{\log(1/\delta)}{\delta^{\frac{1}{2}} \log X}\right).$$

Hence

$$\int_1^{\delta^{-2}} |g(u)|^2 du = O\left(\frac{\log(1/\delta)}{\delta^{\frac{1}{2}} \log X}\right).$$

Next,

$$J(\delta^{-2}, 0)$$

$$\leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k\lambda\mu\nu} \frac{|\beta_k \beta_\lambda \beta_\nu \beta_\mu|}{\nu\lambda} \int_{\delta^{-2}}^{\infty} \exp\left\{-\pi\left(\frac{m^2 k^2}{\lambda^2} + \frac{n^2 \mu^2}{\nu^2}\right) u^2 \sin\delta\right\} du.$$

Since $X = \delta^{-c}$ with $c < \frac{1}{2}$

$$k^2 \lambda^{-2} \sin\delta > AX^{-2}\delta > A\delta^2$$

and

$$\mu^2 \nu^{-2} \sin\delta > AX^{-2}\delta > A\delta^2.$$

As $|\beta_\nu| \leq 1$,

$$\sum_{k\lambda\mu\nu} \frac{|\beta_k \beta_\lambda \beta_\nu \beta_\mu|}{\nu\lambda} = O\left(X^2 \log^2 X\right)$$

so that

$$J(\delta^{-2}, 0) = O\left(X^2 \log^2 X \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{\delta^{-2}}^{\infty} \exp\left\{-A\left(m^2 + n^2\right) u^2 \delta^2\right\} du\right)$$

□

$$= O\left(X^2\log^2 X\int_{\delta^{-2}}^{\infty} e^{-Cu^2\delta^2}du\right)$$
$$= O\left(X^2\log^2 X e^{-C/\delta^2}\right).$$

As $\delta = \frac{1}{X^{1/c}}$ with $0 < c < \frac{1}{8}$, this error term is consumed by the term

$$O\left(\frac{\log 1/\delta}{\delta^{\frac{1}{2}}\log X}\right)$$

so that we may conclude

$$\int_{-\infty}^{\infty}|F(t)|^2dt = O\left(\frac{\log(1/\delta)}{\delta^{\frac{1}{2}}\log X}\right)$$

as desired.

7.1 Additional bounds on $\int F(t)$ and $\int |F(t)|$

The following bounds are useful consequences of 7.1 and 7.2.

Lemma 23. *(10.19)*

$$\int_{-\infty}^{\infty}\left(\int_t^{t+h}|F(u)|du\right)^2dt = O\left(\frac{h^2\log(1/\delta)}{\delta^{\frac{1}{2}}\log X}\right) \tag{7.3}$$

Proof. By Cauchy-Schwarz we have that

$$\int_{-\infty}^{\infty}\left(\int_t^{t+h}|F(u)|du\right)^2dt \leq \int_{-\infty}^{\infty} h\int_t^{t+h}|F(u)|^2dudt.$$

Changing the order of integration yields

$$= h\int_{-\infty}^{\infty}|F(u)|^2du\int_{u-h}^{u}dt = h^2\int_{-\infty}^{\infty}|F(u)|^2du$$

so that the result follows from 7.2. □

Lemma 24. *If $\delta = \frac{1}{T}$, then*

$$\int_0^T|F(t)|dt > AT^{\frac{3}{4}}. \tag{7.4}$$

Proof. Consider the contour integral

$$\left(\int_{\frac{1}{2}+i}^{2+iT} + \int_{2+i}^{2+iT} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}+i} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}+i}\right).$$

Since either $\zeta(s)$ nor $\phi(s)$ have poles in this region, it follows that

$$\left(\int_{\frac{1}{2}+i}^{2+i} + \int_{2+i}^{2+iT} + \int_{2+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}+i}\right)\left(\zeta(s)\phi^2(s)\right) = 0.$$

Let a_n be given by

$$\zeta(s)\phi^2(s) = 1 + \sum_{n=2}^{\infty} \frac{a_n}{n^s}.$$

Since $\phi(s) = \sum \beta_\nu \nu^{-s}$, and $|\beta_\nu| \leq \alpha_\nu^{'}$ where $\alpha_\nu^{'}$ was defined by $\sqrt{\zeta(s)} = \sum \alpha_\nu^{'} \nu^{-s}$, we see that

$$a_n \leq d_2(n).$$

Hence

$$\int_{2}^{2+iT} \zeta(s)\phi^2(s)ds = i\,(T-1) + \sum_{n=2}^{\infty} a_n \int_{2+i}^{2+iT} \frac{ds}{n^s}$$

$$= i(T-1) + O\left(\sum_{n=2}^{\infty} \frac{d_2(n)}{n^2 \log n}\right) = iT + O(1).$$

As $\phi(s) = O\left(X^{\frac{1}{2}}\right)$ for $\sigma \geq \frac{1}{2}$, and $\zeta\left(\frac{1}{2}+iT\right) = O\left(T^{\frac{1}{4}}\right)$, we have

$$\int_{\frac{1}{2}+i}^{2+i} \zeta(s)\phi^2(s)ds = O\left(X\right)$$

and

$$\int_{2+iT}^{\frac{1}{2}+iT} \zeta(s)\phi^2(s)ds = O\left(XT^{\frac{1}{4}}\right).$$

It then follows that

$$\int_0^T \zeta\left(\frac{1}{2}+it\right)\phi^2\left(\frac{1}{2}+it\right)dt \sim T.$$

By definition

$$\int_0^T |F(t)|dt = \int_0^T \frac{1}{\sqrt{2\pi}}\Xi(t)\left(t^2+\frac{1}{4}\right)^{-1} |\phi\left(\frac{1}{2}+it\right)|^2 e^{\left(\frac{1}{4}\pi-\frac{1}{2}\delta\right)t}dt$$

$$= \frac{-1}{2\sqrt{2\pi}}\int_0^T \pi^{-\frac{1}{4}-\frac{1}{2}it}\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)\zeta\left(\frac{1}{2}+it\right)|\phi\left(\frac{1}{2}+it\right)|^2 e^{\left(\frac{1}{4}\pi-\frac{1}{2}\delta\right)t}dt.$$

By Sterlings estimate

$$|\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)| \sim t^{-\frac{1}{4}}e^{-\pi\frac{t}{4}}$$

along with the fact that $\delta = \frac{1}{T}$, it follows that

$$\int_0^T |F(t)|dt > C \int_0^T t^{-\frac{1}{4}} |\zeta \left(\frac{1}{2} + it\right) \phi^2 \left(\frac{1}{2} + it\right)|dt.$$

Hence

$$\int_0^T |F(t)|dt > CT^{-\frac{1}{4}} \left| \int_{\frac{1}{2}T}^T \zeta \left(\frac{1}{2} + it\right) \phi^2 \left(\frac{1}{2} + it\right) dt \right|$$

$$> AT^{\frac{3}{4}}$$

for some positive constant A. □

Lemma 25. *We have that*

$$\int_0^T \left(\int_t^{t+h} |F(u)du\right) dt > AhT^{\frac{3}{4}}. \tag{7.5}$$

Proof. By switching the order of integration, the left hand side becomes

$$\int_0^{T+h} |F(u)|du \int_{\max(0,u-h)}^{\min(T,u)} dt \geq \int_h^T |F(u)|du \int_{u-h}^u dt = h \int_h^T |F(u)|du$$

and the result follows from 7.4. □

8 The Proof

Theorem 26. *There exists a positive constant A such that*

$$N_0(T) > AT \log T.$$

Proof. Let E be the sub-set of $(0, T)$ where

$$\int_t^{t+h} |F(u)|du > \left| \int_t^{t+h} F(u)du \right|.$$

For such values of t, $F(u)$ must change sign in $(t, t+h)$, and hence so must $\Xi(u)$, implying that $\zeta\left(\frac{1}{2} + iu\right)$ has a zero in this interval.

Since $\int_t^{t+h} |F(u)|du$ and $\left| \int_t^{t+h} F(u)du \right|$ are equal except in E, we have that

$$\int_E \int_t^{t+h} |F(u)|dudt \geq \int_E \left(\int_t^{t+h} |F(u)|du - \left| \int_t^{t+h} F(u)du \right|\right) dt$$

$$= \int_0^T \left(\int_t^{t+h} |F(u)|du - \left| \int_t^{t+h} F(u)du \right|\right) dt.$$

Hence by 7.5 we have that

$$\int_E \int_t^{t+h} |F(u)| du dt > A_1 h T^{\frac{3}{4}} - \int_0^T \left| \int_t^{t+h} F(u) du \right| dt.$$

Applying the Cauchy-Schwarz inequality,

$$\int_E \int_t^{t+h} |F(u)| du dt \leq \left((m(E)) \int_E \left(\int_t^{t+h} |F(u)| du \right)^2 dt \right)^{\frac{1}{2}}$$

so that 7.3 with $\delta = \frac{1}{T}$ implies that

$$A_1 h T^{\frac{3}{4}} - \int_0^T \left| \int_t^{t+h} F(u) du \right| dt < A_2 \left(m(E)^{\frac{1}{2}} \right) h T^{\frac{1}{4}} \left(\frac{\log T}{\log X} \right)^{\frac{1}{2}}.$$

Again by the Cauchy Schwarz inequality,

$$\int_0^T \left| \int_t^{t+h} F(u) du \right| dt \leq \left(T \int_0^T \left| \int_t^{t+h} F(u) du \right|^2 dt \right)^{\frac{1}{2}}$$

so that 7.1 implies

$$\int_0^T \left| \int_t^{t+h} F(u) du \right| dt = O\left(\frac{h^{\frac{1}{2}} T^{\frac{3}{4}}}{\log^{\frac{1}{2}} X} \right).$$

Consequently, there are positive contants C_1, C_2 such that

$$C_1 T^{\frac{1}{2}} \left(\frac{\log X}{\log T} \right)^{\frac{1}{2}} - C_2 \frac{T^{\frac{1}{2}}}{h^{\frac{1}{2}} (\log T)^{\frac{1}{2}}} < m(E)^{\frac{1}{2}}.$$

Since $X = \delta^{-c} = T^c$ and $h = (a \log X)^{-1} = (ac \log T)^{-1}$,

$$m(E)^{\frac{1}{2}} > C_1 c^{\frac{1}{2}} T^{\frac{1}{2}} - C_2 (ac)^{\frac{1}{2}} T^{\frac{1}{2}}$$

and by taking a small enough we have that

$$m(E) > C_3 T$$

for some constant C_3. It then follows that of the intervals

$$(0, h),\ (h, 2h),\ (2h, 3h)\ \ldots$$

contained in $(0, T)$ at least

$$[C_3 T/h]$$

must contain points of E. If $(nh, (n+1)h)$ contains a point t of E there must be a zero of $\zeta\left(\frac{1}{2} + iu\right)$ inside $(t, t+h)$ and so in $(nh,\ (n+2)h)$. Allowing for the fact that each zero might be counted twice in this way, there must be at least

$$\frac{1}{2} C_3 T/h > AT \log T$$

zeros in $(0, T)$, and the proof is complete. □

References

[1] E.C. Titchmarsh, *The Theory of the Riemann Zeta Function*, Oxford University

[2] H.L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, Cambridge University Press (2007).

[3] E Lindelöf, , *Quelques remarques sur la croissance de la fonction* $\zeta(s)$, Bull. Sci. Math. 32: 341–356 (1908).