Monday, April 12

started with Kyle's presentation on Turán's method

Recall that given some function $\Delta_0(x)$, we recursively define $\Delta_k(x) = \int_0^x \frac{\Delta_{k-1}(t)}{t}\,dt$.

<u>Lemma</u>: The number of sign changes of $\Delta_{k-1}(x)$ on $[0, A]$ is at least as many as the number of sign changes of $\Delta_k(x)$.

<u>Proof</u>: If we have $x_0 \to 0 < x_1 < x_2 < \cdots < x_N$ where $\mathrm{sign}(\Delta_k(x_j)) = (-1)^j$, then

$$0 < (-1)^j(\Delta_k(x_j) - \Delta_k(x_{j-1})) = \int_{x_{j-1}}^{x_j} \frac{\Delta_{k-1}(t)}{t}\,dt$$

so $(-1)^j \Delta_{k-1}(t) > 0$ for some $x \in [x_{j-1}, x_j]$

We apply this (for example) to

$$\Delta_0(x) = \psi(x) - x = \sum_{n \le x} (\Lambda(n) - 1).$$

$$= -\sum_\rho \frac{x^\rho}{\rho} + \text{error}$$

Then

$$\Delta_k(x) = \sum_{n \le x} (\Lambda(n) - 1)\left(\log\left(\frac{x}{n}\right)\right)^k$$

$$= -\sum_\rho \frac{x^\rho}{\rho^{k+1}} + \text{error}!$$

$$= -\frac{x^{\rho_1}}{\rho_1^{k+1}} + O\left(\sum_{\rho \ne \rho_1} \frac{x^\beta}{|\rho|^{k+1}} + \text{error}\right)$$

where $\rho_1 \approx \frac{1}{2} + i \cdot 14.135$

If RH is true then

implicit constant equals 1!

$$\frac{\Delta_k(x)}{\sqrt{x}} = -\frac{x^{\gamma_1}}{\rho_1^{k+1}} + O\left(\sum_{\rho \ne \rho_1} \frac{1}{|\rho|^{k+1}} + \text{error}\right)$$

If RH is true then

equals 1!

$$\frac{\Delta_k(x)}{\sqrt{x}} = -\frac{x^{i\gamma_1}}{\rho_1^{k+1}} + O\Big(\sum_{\rho\neq\rho_1}' \frac{1}{|\rho|^{k+1}}\Big) \text{ them}$$

In fact for $k=3$,

$$2\,\mathrm{Re}\,\frac{1}{\rho_1^4} \approx \frac{1}{20{,}000} \quad \text{~~while 2Re~~} \quad \sum_{\rho\neq\rho_1}' \frac{1}{\rho^4} \approx \frac{1}{41{,}000}.$$

So if we choose $\{x_j\}$ such that

$$\mathrm{Re}\; x_j^{i\gamma_1} = \cos(\gamma_1 \log x_j) = \pm 1,$$

then we get sign changes of

$$-2\,\mathrm{Re}\,\frac{x^{i\gamma_1}}{\rho_1^4}$$

and hence sign changes of $\frac{\Delta_k(x)}{\sqrt{x}}$ — therefore sign changes of $\Delta(x) = \psi(x) - x$,

sign changes up to $x \le X$ is $\geq \dfrac{\log X}{2\pi/\gamma_1} + O(1)$.

(assuming RH)

Some results from the literature:

- Knapowski, 1985:

Let $W^f(T)$ be the number of sign changes of $f(x)$ in $[0,T]$.

Theorem 1: $W^\pi(T) \gg \log T$

$W^\theta(T) \gg \log T$

constant is ineffective.

Theorem 2: Let $\Theta = \sup(\mathrm{Re}\,\rho)$.

Let $\gamma_0 = \inf(\gamma>0 : \zeta(\Theta + i\gamma) = 0)$.

(note possible $\gamma_0 = \infty$)

Then $\displaystyle\liminf_{T\to\infty} \frac{W^\psi(T)}{\log T} \geq \frac{\gamma_0}{\pi}$.

Schlage-Puchta, 2004:

similar result for # sign changes of

$\pi(x;q,1) - \max_{\substack{(a,q)=1 \\ a \not\equiv 1 \pmod q}} \pi(x;q,a)$

$\left(\gg \frac{kT}{f(q)} \right)$

on GRH.

replace max w/ min

Think about (*) $\pi(x;4,3) - \pi(x;4,1)$.

How many sign changes up to $x \leq X$ do we expect?

Model (*) by a symmetric random walk:

$X_k = \pm 1$ w/ prob $\frac{1}{2}$ each.

$S_k = \sum_{i \leq k} X_k$.

Feller: # sign changes of $\{S_k\}$.

– sign change at S_j if and only if $S_{j-1} S_{j+1} = -1$.

$\Pr\left(\#\{\text{sign changes up to } 2n\} = r \right)$

$= \frac{1}{2^{2n-2}} \binom{2n-1}{n-1-r}$.

$(0 \leq r \leq n-1)$

• Let $C(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$

(CDF for $N(0,1)$), so that

$\Pr(|N(0,1)| < x) = 2C(x) - 1$

As $n \to \infty$,

$\Pr\left(\#\{\text{sign changes up to } 2n\} < x\sqrt{n} \right)$

$\sim 2C(2x) - 1$.

- expected number of sign changes up to N is

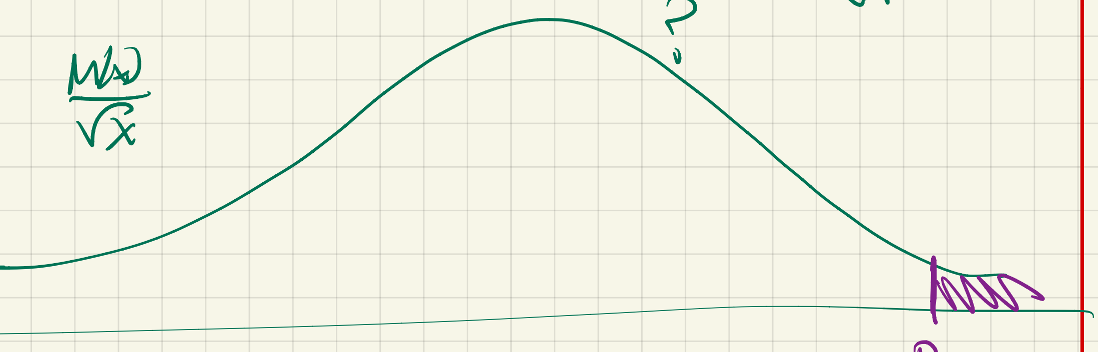
$\sim \sqrt{\frac{2}{\pi}}\sqrt{N} \approx 0.399\sqrt{N}$

- median number of sign changes up to N

$\sim 0.337\sqrt{N}$.

Maybe then we conjecture:

$\#\{$sign changes of $\pi(x;4,3)-\pi(x;4,1)$ up to $X\} \asymp \sqrt{\frac{X}{\log x}}$?



$\Pr(N(0,1)>p) \asymp \frac{e^{-p^2/2}}{p}$

sample $e^{p^2/2}$ times to get $v_0)>p$

x $\sqrt{\log(\sqrt{2x}}$