Wednesday, April 14

started with Devang's presentation on Number field races

Hooley's conjecture:

Notation:

$$G(x;q) = \sum_{(a,q)=1} \left| \theta(x;q,a) - \frac{x}{\phi(q)} \right|^2$$

$$V(x;q) = \sum_{(a,q)=1} \left| \psi(x;q,a) - \frac{1}{\phi(q)} \sum_{\substack{n\le x \\ (n,q)=1}} \Lambda(n) \right|^2$$

If we use orthogonality of Dirichlet characters (Parseval's identity):

$$G(x;q) = \frac{1}{\phi(q)} \sum_{\chi (\mathrm{mod}\, q)} \left| \theta(x,\chi) - \begin{cases} x, & \text{if } \chi = \chi_0, \\ 0, & \text{otherwise} \end{cases} \right|^2$$

$$V(x;q) = \frac{1}{\phi(q)} \sum_{\chi (\mathrm{mod}\, q)} |\psi(x;\chi)|^2.$$

(recall: $\theta(x,\chi) = \sum_{p\le x} \chi(p) \log p$, $\quad \psi(x;\chi) = \sum_{n\le x} \Lambda(n)\chi(n)$)

(recall also: $\psi(x,\chi) - \begin{cases} x, & \text{if } \chi = \chi_0, \\ 0, & \text{else} \end{cases} =$

$$= - \sum_{\substack{\rho \text{ non-trivial} \\ \text{zero of } L(s,\chi)}} \frac{x^\rho}{\rho} + \text{error} \Bigg)$$

- Hooley (1970s) conjectured as soon as q tends to infinity with x, then $G(x;q) \ll x \log q$,
$V(x;q) \ll x \log q$.
- we don't expect this upper bound when q is fixed
- for example, when $q = 4, 3$, Littlewood's results on $\pi(x;4,3) - \pi(x;4,1)$ and $\pi(x;3,2) - \pi(x;3,1)$ shows

$$\begin{matrix} G(x,4) \\ V(x,3) \end{matrix} = \Omega\left(x(\log\log\log x)^2\right).$$

- Davidoff (80s?) showed for fixed q,

$$G(x;q),\ V(x;q) = \Omega\left(\frac{1}{\phi(q)} x (\log\log\log x)^2\right)$$

- Hooley (1970s) conjectured as soon as q tends to infinity with x, then $G(x;q) \ll x \log q$, $V(x;q) \ll x \log q$.
- The average version $\frac{1}{Q} \sum_{q \le Q} V(x;q)$, ✲ is the Barban–Davenport–Halberstam theorem: ✲ $\ll x \log Q$ when $Q > x/(\log x)^A$. On GRH, ✲ $\ll x \log Q$ for $Q > x^{\frac{1}{2}+\varepsilon}$.
- For individual q: if we assume GRH + strong version of prime pairs conjecture (Hardy–Littlewood), Friedlander–Goldston showed $V(x;q) \ll x \log q$ for $q \ge x^{\frac{1}{2}+\varepsilon}$.
- Nothing known, even conditionally, for $q < x^{\frac{1}{2}}$.

Fiorilli, "The distribution of the variance of primes in arithmetic progressions" (2013/14?).

Fiorilli conjectured that

$G(x;q)$, $V(x;q)$ $\ll x \log q$ for $q > (\log\log x)^{1+\delta}$

Heuristic/motivation: $V(x;q)$ ✲ should have the same limiting distribution as some random variable H_q.

$\mathbb{E}(H_q) \sim \phi(q) \log q$

$\sigma^2(H_q) \sim 2\phi(q)(\log q)^2$.

H_q "roughly normal"

✲ $\phi(q)$? $\frac{V(x;q)}{x}$

Fiorilli proved, about H_q, that

$$\frac{1}{4}e^{-c_1\varepsilon^2\phi(q)} \le \Pr(|H_q - \phi(q)\log q| > \varepsilon\,\phi(q)\log q) \le 2e^{-c_2\varepsilon^2\phi(q)}$$

- tail estimates for H_q / large deviations

$\Rightarrow$ (heuristically) that $V(x;q) \ll x\log q$ should hold "with probability 1" even if q is as small as $(\log\log x)^{1+\delta}$.

- some heuristic suggests that $V(x;q) \not\ll x\log q$ when $q < (\log\log x)$.
  - Hooley's conjecture should be false (for very very small q)

Fiorilli–M, "Disproving Hooley's conjecture", 2021+.

Theorem: * $G(x;q)$, $V(x;q)$ $= \Omega\left(x\log q\,\frac{\log\log x}{q}\right)$

In particular, Hooley's conjecture cannot hold in the range $q < \delta\log\log x$ if δ is small enough.

* for a positive proportion of q

↳ if GRH true, holds for all moduli q.

- GRH false. Let ρ_1 be a zero of some $L(s,\chi_1)$ mod q_1, with $\beta_1 = \mathrm{Re}\,\rho_1 > \frac{1}{2}$.

Then we can show if χ (mod q) is induced by χ_1 (mod q_1) then

$$V(x;q) \ge \frac{1}{\phi(q)}|\psi(x;\chi)|^2 \gg x^{2\beta_1-\varepsilon}$$

- we need $q_1 \mid q$.