Wednesday, April 7

Started with Elchin's presentation on Biases in the Distribution of Consecutive Primes

MV Section 5.1

Warm-up calculation:

- $$D^{\psi}(x) = \int_0^x \psi(t)\,dt$$
$$= \int_0^x \sum_{n\le t} \Lambda(n)\,dt$$
$$= \sum_{n\le x} \Lambda(n) \int_n^x dt = \sum_{n\le x} \Lambda(n)(x-n).$$

- $$D^{\psi}(x) = \int_0^x \left(\frac{1}{2\pi i} \int_{(c)} -\frac{\zeta'}{\zeta}(s) \frac{t^s}{s}\,ds \right) dt$$
$$= \frac{1}{2\pi i} \int_{(c)} -\frac{\zeta'}{\zeta}(s) \left(\int_0^x \frac{t^s}{s}\,dt \right) ds$$
$$= \frac{1}{2\pi i} \int_{(c)} -\frac{\zeta'}{\zeta}(s) \frac{t^{s+1}}{s(s+1)}\,ds$$

- $$D^{\psi}(x) = \int_0^x \left(t - \sum_\rho \frac{t^\rho}{\rho} + \text{error} \right) dt$$
$$= \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} + \int \text{error}.$$

More general:

- any sequence $\{a_n\}$.
- $$D_0(x) = \sum_{n\le x} a_n$$
$$\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

Perron's formula: $$D_0(x) = \frac{1}{2\pi i} \int_{(c)} \alpha(s) \frac{x^s}{s}\,ds$$

- $$D_k(x) = \int_0^x D_{k-1}(t)\,dt.$$

We can check:

$$D_k(x) = \sum_{n\le x} a_n \frac{(x-n)^k}{k!}$$

also

$$D_k(x) = \frac{1}{2\pi i} \int_{(c)} \alpha(s) \frac{x^{s+k}}{s(s+1)\cdots(s+k)}\,ds.$$

A similar smoothing/averaging scheme:

- $\{a_n\}$, $\alpha(s)$ as before

$$\Delta_0(x) = \sum_{n \le x} a_n.$$

- Now set $\Delta_k(x) = \int_0^x \frac{\Delta_{k-1}(t)}{t}\,dt$

(logarithmic smoothing)

We can check:

$$\Delta_k(x) = \frac{1}{k!}\sum_{n \le x} a_n \left(\log \frac{x}{n}\right)^k$$

and also

$$\Delta_k(x) = \frac{1}{2\pi i}\int_{(c)} \alpha(s)\,\frac{x^s}{s^{k+1}}\,ds$$

One other scheme:

- $\{a_n\}$, $\alpha(s)$ as before

$$P(x) = \sum_{n=1}^{\infty} a_n e^{-nx}.$$

(approximately cutting off the sum at $n = \frac{1}{x}$)

(think of taking $x \to 0+$)

It turns out that

$$P(x) = \frac{1}{2\pi i}\int_{(c)} \alpha(s)\,\Gamma(s)\,x^{-s}\,ds$$

(Γ is Euler Gamma function)

In particular,

$$\Delta_k^{\psi}(x) = \frac{1}{k!}\sum_{n \le x} \Lambda(n)\left(\log \frac{x}{n}\right)^k$$

$$= x - \sum_{\rho} \frac{x^{\rho}}{\rho^{k+1}} + \text{error}.$$

On Monday we'll look at:

- behaviour/distribution of $-\sum_{\rho} \frac{x^{\rho}}{\rho^{k+1}}$
- relate sign changes of $\Delta_k^{\psi}(x) - x$ to sign changes of $\psi(x) - x$.