Wednesday February 10

- Started with Elchin's presentation on Pólya's Conj.

Recall from Monday: if $\chi \pmod q$ is primitive, $q > 1$, then

$$\psi(x,\chi) = \sum_{n \le x} \Lambda(n)\chi(n),$$

$$= -\sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ \text{(nontrivial zeros)}}} \frac{x^\rho}{\rho} + O_q(\log x)$$

We also know

$$\psi(x;q,a) = \sum_{\substack{n \le x \\ n \equiv a \pmod q}} \Lambda(n)$$

$$= \frac{1}{\phi(q)} \sum_{\chi \pmod q} \psi(x,\chi)\overline{\chi}(a).$$

Sidenote: what if $\chi \pmod q$ is induced by $\chi^* \pmod{q^*}$?

$$\psi(x,\chi^*) = \sum_{\substack{n \le x \\ (n,q^*)=1}} \chi^*(n)\Lambda(n)$$

$$= \underbrace{\sum_{\substack{n \le x \\ (n,q)=1}} \chi(n)\Lambda(n)}_{\psi(x,\chi).} + \sum_{\substack{n \le x \\ (n,q)>1 \\ (n,q^*)=1}} \chi^*(n)\Lambda(n)$$

$$\psi(x,\chi^*) - \psi(x,\chi) = \sum_{\substack{n \le x \\ (n,q)>1 \\ (n,q^*)=1}} \chi^*(n)\Lambda(n)$$

$$= \sum_{\substack{p \mid q \\ p \nmid q^*}} \log p \cdot \sum_{\substack{r \in \mathbb{N} \\ p^r \le x}} \chi^*(p^r)$$

$$\ll \sum_{p \mid q} \log p \cdot \sum_{r \le \frac{\log x}{\log p}} 1$$

$$\le \sum_{p \mid q} \log p \cdot \left\lfloor \frac{\log x}{\log p} \right\rfloor \ll \log x \cdot \log q.$$

So we get an explicit formula for

$$\psi(x;q,a) = \frac{x}{\phi(q)} - \frac{1}{\phi(q)}\sum_{\chi (\mathrm{mod}\, q)} \overline{\chi}(a) \sum_{\substack{\rho \\ L(\rho,\chi)=0}} \frac{x^\rho}{\rho} + O(\log x \log q).$$ $\ast$

(main term of $\psi(x,\chi_0)$)

Note: $\frac{x^\rho}{\rho} = x^\beta \cdot \frac{x^{i\gamma}}{\rho} = x^\beta \frac{e^{i\gamma \log x}}{\rho}$.

$\frac{e^{i\gamma \log x}}{\rho}$ β, γ a function of $\log x$, uniformly distributed (in the limit) on the circle with centre 0 and radius $1/|\rho|$ & $1/|\gamma|$.

We might expect, from $\ast$, (assuming GRH)

$\frac{\psi(x;q,a) - x/\phi(q)}{\sqrt{x}}$ might have a symmetric limiting logarithmic distribution function.

Recall $\psi(x) = \sum_{n \le x} \Lambda(n) = \sum_{\substack{\text{primes } p \\ r \in \mathbb{N} \\ p^r \le x}} \log p$

$\theta(x) = \sum_{p \le x} \log p$

$\psi(x) = \theta(x) + \theta(x^{\frac{1}{2}}) + \theta(x^{\frac{1}{3}}) + \dots$

Therefore $= \sum_{r=1}^{\infty} \theta(x^{\frac{1}{r}})$.

$$\theta(x) = \sum_{r=1}^{\infty} \mu(r)\psi(x^{\frac{1}{r}})$$

$= \psi(x) - \psi(x^{\frac{1}{2}}) - \psi(x^{\frac{1}{3}}) - \psi(x^{\frac{1}{5}}) + \psi(x^{\frac{1}{6}}) \dots$

$= \psi(x) - \psi(x^{\frac{1}{2}}) + O(x^{\frac{1}{3}})$.

$= \psi(x) - (x^{\frac{1}{2}} + \text{error}) + \text{error}$

$$\frac{\theta(x) - x}{\sqrt{x}} = \frac{\psi(x) - x}{\sqrt{x}} - 1 + o(1).$$

$$= -\sum_{\substack{\rho \\ \zeta(\rho)=0}} \frac{x^\rho}{\rho} - 1 + o(1).$$

Similarly $\pi(x) = \int_{2^-}^{x} \frac{1}{\log t}\, d(\theta(t))$

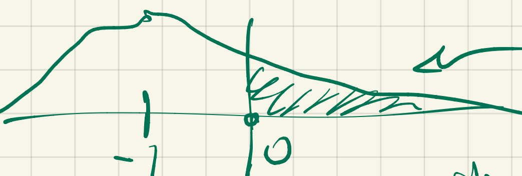
$$= \frac{\theta(x)}{\log x} + \int_{2}^{x} \frac{\theta(t)}{t \log^2 t}\, dt.$$

"We expect"

$\frac{\pi(x) - li(x)}{\sqrt{x}/\log x}$ to have a limiting logarithmic distribution function that is symmetric around -1.

$\Rightarrow$ $\pi(x) < li(x)$ is more likely than $\pi(x) > li(x)$.

$\Rightarrow$ if the distribution looks like



$\leftarrow$ we expect $\pi(x) > li(x)$ does occur with a positive logarithmic density.

$$\Pi(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \dots$$

For primes in APs,

$$\psi(x;q,a) = \sum_{\substack{p \le x \\ p \equiv a \,(\mathrm{mod}\, q)}} \log p + \sum_{\substack{p^2 \le x \\ p^2 \equiv a \,(\mathrm{mod}\, q)}} \log p + O(x^{\frac{1}{3}})$$

$$= \theta(x;q,a) + \sum_{\substack{b \,(\mathrm{mod}\, q) \\ b^2 \equiv a \,(\mathrm{mod}\, q)}} \sum_{\substack{p \le \sqrt{x} \\ p \equiv b \,(\mathrm{mod}\, q)}} 1 + O()$$

$$= \theta(x;q,a) + \sum_{\substack{b \,(\mathrm{mod}\, q) \\ b^2 \equiv a \,(\mathrm{mod}\, q)}} \theta(\sqrt{x};q,b) + O(x^{\frac{1}{3}}).$$

Thus

$$\frac{\theta(x;q,a) - \frac{x}{\phi(q)}}{\sqrt{x}} = \frac{\psi(x;q,a) - x/\phi(q)}{\sqrt{x}} - \frac{1}{\phi(q)} \#\{b \,(\mathrm{mod}\, q) : b^2 \equiv a \,(\mathrm{mod}\, q)\} + o(1).$$

$$\begin{array}{ccc} \psi & \to & \theta \\ \downarrow & & \downarrow \\ \Pi & \to & \pi \end{array}$$