Monday, February 22

Start: Devang's presentation on Dedekind ζ functions

Explicit formula for $\psi(x;q,a) = \sum_{\substack{n\le x \\ n\equiv a \pmod q}} \Lambda(n)$

(always $(a,q)=1$):

$$\psi_0(x;q,a) = \frac{1}{2}\left(\psi(x^+;q,a) + \psi(x^-;q,a)\right)$$

$$= \frac{x}{\phi(q)} - \frac{1}{\phi(q)} \sum_{\chi \pmod q} \bar{\chi}(a) \sum_{\rho} \frac{x^\rho}{\rho}$$

$|\mathrm{Im}\,\rho| \le T$ — nontrivial zeros of $L(s,\chi)$

$$+ \sum_{\chi \pmod q} \bar{\chi}(a)\left(\frac{1}{2}\log(x-1) - \frac{\chi(-1)}{2}\log(x+1) + C(\chi)\right)$$

$$+ \text{error}_q(x,T).$$

- zero-free* region for $L(s,\chi)$, of the shape $\sigma > 1 - \frac{c}{\log qT}$; has us choose $T \approx \exp(c\sqrt{\log x})$.

- suppose some quadratic $\chi_1 \pmod q$ has an exceptional zero $\beta_1 \in \mathbb{R}$ very close to $s=1$. Then two "error" terms are actually big:
  - since $L(\beta_1,\chi_1)=0$, also $L(1-\beta_1,\chi_1)=0$ by functional equation; so $\frac{x^{1-\beta_1}}{1-\beta_1}$ could be big.
  - $C(\chi) = \frac{L'}{L}(1,\bar{\chi}) + \text{small}\ldots$; $\frac{1}{L(1,\bar{\chi}_1)}$ could be big.

Turns out: the sum of these contributions is very small (equation (12.11) in MV).

$$\psi_0(x;q,a) = \frac{x}{\phi(q)} + O\left(x\exp(-c\sqrt{\log x})\right)$$

if there is no exceptional zero.

- implicitly involves $q \le \exp(c\sqrt{\log x})$.

Corollary 11.17 (Page) Suppose $(a,q)=1$.

Then $\psi(x;q,a) = \frac{x}{\phi(q)} + O(x\exp(-c\sqrt{\log x}))$

if there is no exceptional zero; if β_1 is an exceptional zero then

$$\psi(x;q,a) = \frac{x}{\phi(q)} - \frac{\chi_1(a)}{\phi(q)}\frac{x^{\beta_1}}{\beta_1} + O(x\exp(-c\sqrt{\log x}))$$

Remember: • $1-\beta_1 \gg (q^{\frac{1}{2}}\log^2 q)^{-1}$.

• Siegel's theorem: $1-\beta_1 \gg_\varepsilon q^{-\varepsilon}$ (ineffectively).

Thought experiment: q is prime.

$\chi_1(n) = \left(\frac{n}{q}\right)$. Suppose $L(\beta_1,\chi_1)=0$ for

$\beta_1 = 1 - \frac{1}{\sqrt{q}}$

- If $\left(\frac{a}{p}\right)=-1$: $\frac{1}{\phi(q)}\left(x + \frac{x^{\beta}}{\beta}\right) \approx \frac{2x}{\phi(q)}$.

- Suppose $\chi_1(a) = \left(\frac{a}{p}\right) = +1$.

Main term is $\frac{1}{\phi(q)}\left(\frac{x^1}{1} - \frac{x^{\beta_1}}{\beta_1}\right)$.

mean value theorem applied to $f(t) = \frac{x^t}{t}$:

$\frac{x^1}{1} - \frac{x^{\beta_1}}{\beta_1} = (1-\beta_1)f'(\xi)$ for some $\xi \in (\beta_1, 1)$

$f'(t) = \frac{x^t}{t^2}(t\log x - 1)$

$\frac{1}{\xi} = 1 + O(1-\xi)$

$x^\xi = x \cdot e^{(1-\xi)\log x}$

$= x(1+O((1-\xi)\log x))$

$f'(\xi) = x(\log x + O(1))(1+O((1-\xi)\log x))$.

Roughly, main term is $\frac{1}{\phi(q)} x(1-\beta_1)\log x$.

$= \frac{1}{\phi(q)} x \frac{\log x}{\sqrt{q}}$.

• If $q, x \to \infty$ at a rate where $\log x = o(\sqrt{q})$, then this main term is o(what it should be).

is $q^{\frac{1}{2}-\varepsilon}$

$x \approx e^{q^{\frac{1}{2}-\varepsilon}}$

$\Leftrightarrow q > (\log x)^{2+\varepsilon}$.

Corollary 11.19 (Siegel-Walfisz theorem)

For any $A>0$, if $(a,q)=1$ and $q \le (\log x)^A$, then

$$\psi(x;q,a) = \frac{x}{\phi(q)} + O_A\left(x\exp(-c\sqrt{\log x})\right).$$

constant independent of q — but ineffective

Thus

$$\theta(x;q,a) = (\text{same thing})$$

$$\pi(x;q,a) = \frac{li(x)}{\phi(q)} + O_A\left(x\exp(-c\sqrt{\log x})\right)$$

Linnik's theorem: There is an absolute constant L such that $\forall q$ there exists a prime $p \equiv a \pmod q$, $p < q^L$.

- Pan (1958) $L = 5{,}448$
- Heath-Brown (1992) $L = 5.5$ (effective)
- Xylouris (2011) $L = 5$
- GRH $\Rightarrow L \le 2+\varepsilon$ · conjecture: $L = 1+\varepsilon$.