Wednesday February 24

Reminder of "Landau's theorem":

(Theorem 1.7) Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$.

If $a_n \geq 0$ for all sufficiently large n, then $\alpha(s)$ has a singularity at $s = \sigma_c$ (its abscissa of convergence).

Variant:

Lemma 15.1: Let $A(x)$ be a bounded, Riemann-integral function, and suppose $A(x) \geq 0$ for $x > X_0$. Let σ_c be the infimum of $\sigma \in \mathbb{R}$ such that $\int_1^\infty A(x) x^{-\sigma} dx$ converges. Then the Dirichlet integral

$$F(s) = \int_1^\infty A(x) x^{-s} dx$$

is analytic in $\{\sigma > \sigma_c\}$ but not at $s = \sigma_c$.

In usage:

if $A(x)$ is eventually nonnegative, then the rightmost singularity of $F(s)$ is on the real axis.

Contrapositive: If the rightmost singularity of $F(s)$ is not on the real axis, then $A(x)$ is not eventually nonnegative (or eventually nonpositive). In other words, $A(x)$ changes sign infinitely often.

Recall:

$$\psi(x) = \sum_{n \leq x} \Lambda(n) \qquad \theta(x) = \sum_{p \leq x} \log p$$

$$\Pi(x) = \sum_{n \leq x} \frac{\Lambda(n)}{\log n} \qquad \pi(x) = \sum_{p \leq x} 1$$

Let Θ denote the supremum of the real parts of the zeros of $\zeta(s)$.

So $\frac{1}{2} \leq \Theta \leq 1$. RH $\iff \Theta = \frac{1}{2}$.

Theorem 15.2: For every $\varepsilon > 0$,

$$\psi(x) - x = \Omega_{\pm}(x^{\Theta-\varepsilon}) \quad \text{and}$$

$$\pi(x) - \mathrm{li}(x) = \Omega_{\pm}(x^{\Theta-\varepsilon}).$$

(In other words, $\limsup_{x\to\infty} \frac{\psi(x)-x}{x^{\Theta-\varepsilon}} > 0$

and $\liminf_{x\to\infty} \frac{\psi(x)-x}{x^{\Theta-\varepsilon}} < 0$.)

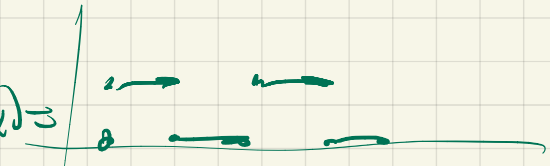
Proof: We know $-\frac{\zeta'}{\zeta}(s) = s\int_1^\infty \psi(x) x^{-s-1}\,dx$.

$$\left[-\frac{\zeta'}{\zeta}(s) = \int_{1^-}^\infty x^{-s}\, d\psi(x)\right]$$

So $\int_1^\infty (\psi(x) - x - x^{\Theta-\varepsilon}) x^{-s-1}\,dx$

$$= \underbrace{-\frac{1}{s}\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}} - \frac{1}{s-\Theta+\varepsilon}.$$

- no singularity at $s=1$
- by definition of Θ, there are nonreal singularities with $\sigma > \Theta - \varepsilon$
- analytic on the real axis for $\sigma > \Theta - \varepsilon$.

Eg $A(x) =$ 

Then $\int_{1^-}^\infty x^{-s}\, dA(x) = \zeta(s)(1-2^{1-s})$

$$\frac{1}{s}\zeta(s)(1-2^{1-s}) = \int_1^\infty A(x) x^{-s-1}\,dx$$

Since the rightmost singularity is not on the real axis, we conclude that $\psi(x) - x - x^{\Theta-\varepsilon}$ is not eventually nonnegative, that is, $\psi(x) - x = \Omega_-(x^{\Theta-\varepsilon})$.

Similarly, replacing $\psi(x) - x - x^{\Theta-\varepsilon}$ with $\psi(x) - x + x^{\Theta-\varepsilon}$ shows that $\psi(x) - x = \Omega_+(x^{\Theta-\varepsilon})$.

Similarly, one can show

$$s\int_2^\infty li(x)x^{-s-1}dx = -\log(s-1) + r(s)$$

where $r(s)$ is entire, and then

$$\int_2^\infty (x^{\Theta-\varepsilon} - (\Pi(x) - li(x)))\, x^{-s-1}dx = \frac{1}{s-\Theta+\varepsilon} - \frac{1}{s}\log(\zeta(s)(s-1)) + \frac{r(s)}{s} \ldots$$

From this, and

$$\theta(x) = \psi(x) - x^{\frac{1}{2}} + O(x^{\frac{1}{2}}\exp(-c\sqrt{\log x}))$$

$$\pi(x) = \Pi(x) - \frac{x^{\frac{1}{2}}}{\log x} + O(x^{\frac{1}{2}}/\log^2 x)$$

we can conclude:

- $\theta(x) - x = \Omega_-(x^{\Theta-\varepsilon})$

  $\pi(x) - li(x) = \Omega_-(x^{\Theta-\varepsilon})$;

- if $\Theta > \frac{1}{2}$, then

  $\theta(x) - x = \Omega_+(x^{\Theta-\varepsilon})$

  $\pi(x) - li(x) = \Omega_+(x^{\Theta-\varepsilon})$.

Theorem 18.3: Suppose ρ is a nontrivial zero of $\zeta(s)$ with $\mathrm{Re}\,\rho = \Theta$. Then

$$\limsup_{x\to\infty} \frac{\psi(x)-x}{x^\Theta} \geq \frac{1}{|\rho|}$$

$$\liminf_{x\to\infty} \frac{\psi(x)-x}{x^\Theta} \leq -\frac{1}{|\rho|}.$$

Corollary 18.4: $\psi(x) - x = \Omega_{\pm}(\sqrt{x})$

$\theta(x) - x = \Omega_{-}(\sqrt{x})$.

Littlewood (1916):

$$\psi(x) - x = \Omega_\pm(x^{\frac{1}{2}}\log\log\log x).$$

$$\Pi(x) - x = \Omega_\pm\left(\frac{x^{\frac{1}{2}}}{\log x}\log\log\log x\right).$$

$$\Downarrow$$

$$\theta(x) - x = \Omega_\pm(x^{\frac{1}{2}}\log\log\log x)$$

$$\pi(x) - li(x) = \Omega_\pm\left(\frac{x^{\frac{1}{2}}}{\log x}\log\log\log x\right)$$

$\Rightarrow$ $\pi(x) - li(x)$ changes sign infinitely often.

How would we apply these ideas to primes in arithmetic progressions?

for example: $\psi(x;q,a) - \psi(x;q,b)$.

$$\int x^{-s}\, d\left(\psi(x;q,a) - \psi(x;q,b)\right) =$$

$$\frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \left(\bar{\chi}(b) - \bar{\chi}(a)\right) \frac{L'}{L}(s,\chi). \quad (*)$$

(in place of $-\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}$)

So the relevant number, analogous to Θ, is the supremum of the real parts of singularities of $(*)$.

- sort of: sup(real parts(zeros of all the $L(s,\chi)$))
- but: some $L(s,\chi)$ don't appear in the sum (when $\chi(a) = \chi(b)$).
- there might be cancellation of singularities in different terms of the sum.
- maybe $L(s,\chi)$ have real nontrivial zeros.