Wednesday, February 3

Warm-up inequality: for any quadratic character $\chi \pmod q$, and $\sigma > 1$,

$$-\frac{L'}{L}(\sigma,\chi_0) - \frac{L'}{L}(\sigma,\chi) = \sum_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{\Lambda(n)}{n^\sigma}\left(1+\chi(n)\right) \geq 0. \qquad (W)$$

Lemma 11.2: For any character $\chi \pmod q$, and any $\sigma > 1$, $t \in \mathbb{R}$,

$$\mathrm{Re}\Big(\underbrace{-3\frac{L'}{L}(\sigma,\chi_0) - 4\frac{L'}{L}(\sigma+it,\chi) - \frac{L'}{L}(\sigma+2it,\chi^2)}_{(*)}\Big) \geq 0.$$

Proof: The left-hand side equals

$$\mathrm{Re}\sum_{\substack{n\geq 1\\(n,q)=1}} \frac{\Lambda(n)}{n^\sigma}\left(3+4\frac{\chi(n)}{n^{it}}+\frac{\chi^2(n)}{n^{2it}}\right).$$

Write $\frac{\chi(n)}{n^{it}} = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

So this equals

$$\sum_{\substack{n\geq 1\\(n,q)=1}} \frac{\Lambda(n)}{n^\sigma}\left(3+4\cos\theta+\cos 2\theta\right)$$

$$= \sum_{\substack{n\geq 1\\(n,q)=1}} 2(1+\cos\theta)^2 \geq 0. \quad //$$

Consequence: $L(s,\chi) \neq 0$ when $\sigma = 1$.

Why?

- As $\sigma \to 1^+$, $L(\sigma,\chi_0)$ has a simple pole, so $-\frac{L'}{L}(\sigma,\chi_0)$ has a simple pole of residue 1. Thus $-3\frac{L'}{L}(\sigma,\chi_0) \sim \frac{3}{\sigma-1}$ as $\sigma \to 1^+$.
- If $L(1+it,\chi) = 0$, then $-\frac{L'}{L}(\sigma+it,\chi)$ has a simple pole of residue -1 as $\sigma \to 1^+$; thus $-4\frac{L'}{L}(\sigma+it,\chi) \sim \frac{-4}{(\sigma+it)-(1+it)}$ as $\sigma \to 1^+$.
- $-\frac{L'}{L}(\sigma+2it,\chi^2)$ is bounded above since there's no pole. (as long as $\chi^2 \neq \chi_0$, or $t \neq 0$)

Some tools from complex analysis, about analytic functions f on closed discs:

- "Jensen's lemma" (Lemma 6.1, MV): upper bound for the number of zeros of f in a smaller disc, in terms of $|f(0)|$ and $\max|f(z)|$ (and the discs' radii)
- "Borel-Carathéodory lemma" (Lemma 6.2): upper bound for $|f|$ and $|f'|$ in a smaller disc when $f(0)=0$, in terms of $\max(\operatorname{Re} f(z))$ (and the discs' radii).

<u>Lemma 6.3</u>: Suppose $f(z)$ is analytic on $|z|\le 1$, that $|f(z)|\le M$ there, and $f(0)\ne 0$. Fix $0<r<R<1$. Then, for $|z|\le r$,

$$\frac{f'}{f}(z) = \sum_{k=1}^{K} \frac{1}{z-z_k} + O_{R,r}\left(\log \frac{M}{|f(0)|}\right),$$

where $z_1,\dots,z_K$ are the zeros of $f(z)$ on the medium disc $|z|\le R$.

Notes: • if f has a multiple zero, then we just list it multiple times.

• if f is a polynomial, then $\frac{f'}{f}(z)=\sum \frac{1}{z-z_k}$ exactly.

Consequences:

<u>Lemma 6.4</u>: If $|t|\ge\frac{7}{8}$ and $\frac{5}{6}\le\sigma\le 2$, then

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \frac{1}{s-\rho} + O(\underline{\log \cancel{t}}\,\tau)$$

where the sum is over zeros, $\rho=\beta+i\gamma$, of $\zeta(s)$ for which $\left|\rho-\left(\frac{3}{2}+it\right)\right|\le\frac{5}{6}$.

- not really $\log t$; we need something like $\max\{\log|t|,1\}$ or $\log(|t|+4)$.

MV write $\tau=|t|+4$.

This result for $\zeta(s)$ implies an analogous result for $L(s,\chi_0)$

Lemma 11.1 for $\chi_0 \pmod q$:

When $\frac{5}{6} \le \sigma \le 2$,

$$-\frac{L'}{L}(s,\chi_0) = \frac{1}{s-1} - \sum_{\rho} \frac{1}{s-\rho} + O(\log(q\tau))$$

where ρ runs over zeros of $L(s,\chi_0)$ satisfying $|\rho - (\frac{3}{2}+it)| \le \frac{5}{6}$.

(uses $L(s,\chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{\chi(p)}{p^s}\right)$.)

It also follows from Lemma 6.3:

Lemma 11.1 for nonprincipal χ:

same statement without the $\frac{1}{s-1}$.

Note: if $s = \sigma + it$ with $\sigma > 1$, and $\rho = \beta + i\gamma$ has $\beta < 1$ (as it must), then $\mathrm{Re}(s-\rho) > 0$.

$$\frac{1}{\mathrm{Re}(s-\rho)} > 0.$$

Hence for upper bounds on $\mathrm{Re}(-\frac{L'}{L}(s,\chi))$, we can throw some or all ρ away:

$$\mathrm{Re}\left(-\frac{L'}{L}(s,\chi)\right) \le O(\log q\tau)$$
$(\chi \ne \chi_0)$

or, if ρ_1 is a special zero near s, then

$$\mathrm{Re}\left(-\frac{L'}{L}(s,\chi)\right) \le -\mathrm{Re}\frac{1}{s-\rho} + O(\log q\tau).$$

Theorem 11.3 : "zero-free" region for $L(s,\chi)$:

There exists an absolute constant $c>0$ such that, for any Dirichlet character $\chi \pmod q$, the region

$$\left\{ s=\sigma+it : \sigma > 1 - \frac{c}{\log q\tau} \right\}$$

contains no zeros of $L(s,\chi)$ unless χ is quadratic, in which case there might be one zero $\rho_1 = \beta_1 + 0i$ on the real axis (in which case ~~$\beta_1 < 0$~~). "exceptional zero" $\beta_1 < 1$.

Sketch of proof:

Case 1: χ complex. Let $\rho_0 = \beta_0 + i\gamma_0$ be a zero of $L(s,\chi)$. Look at $s = 1+\delta+i\gamma_0$, for some $0<\delta\le 1$.

By Lemma 11.7 (throwing away ρ to get an upper bound):

- $\mathrm{Re}\left(-\frac{L'}{L}(1+\delta,\chi_0)\right) \le \frac{1}{\delta} + O(\log q\tau)$.
- $\mathrm{Re}\left(-\frac{L'}{L}(1+\delta+i\gamma_0,\chi)\right) \le -\frac{1}{1+\delta-\beta_0} + O(\log q\tau)$
- $\mathrm{Re}\left(-\frac{L'}{L}(1+\delta+2i\gamma_0,\chi^2)\right) \le O(\log q\tau)$.

Hence

$$0 \le \mathrm{Re}\left(-3\frac{L'}{L}(1+\delta,\chi_0) - 4\frac{L'}{L}(1+\delta+i\gamma_0,\chi) - \frac{L'}{L}(1+\delta+2i\gamma_0,\chi^2)\right)$$

$$\le \frac{3}{\delta} - \frac{4}{1+\delta-\beta_0} + O(\log(q\gamma_0)).$$

Incompatible, as $\delta \to 0^+$, with $1-\beta_0$ being really small.

We get a bound of the form

$$1-\beta_0 \ge \frac{c}{\log q\gamma_0}.$$

What do we need to modify if χ is quadratic ($\chi^2 = \chi_0$)?

- If $|\gamma_0|$ isn't too small, then we're not close to the pole of $L(s,\chi_0)$, and the argument goes through.
- If $|\gamma_0|$ is small, then we have

$$\underset{L(s,\chi_0)}{3 \text{ pole votes}} - \underset{L(s,\chi)}{\cancel{4}\ 8 \text{ zero votes}} + \underset{L(s,\chi^2)}{1 \text{ pole vote}}$$

Trick: zero $\rho_0 = \beta_0 + i\gamma_0$ also causes a zero $\bar{\rho}_0 = \beta_0 - i\gamma_0$ since χ is real. – as long as $\gamma_0 \neq 0$.

- When χ is quadratic, concerning real zeros: if β_1, β_2 are two real zeros, then 3 pole votes − 8 zero votes + 1 pole vote.

Theorem 11.4 Let $\chi \pmod q$ be nonprincipal. Suppose that

$$\sigma \geq 1 - \frac{c/2}{\log q\tau}.$$

- If $L(s,\chi)$ has no exceptional zero, or if β_1 is exceptional but $|s-\beta_1| \geq \frac{1}{\log q}$, then

$$\frac{L'}{L}(s,\chi) \ll \log q\tau,$$

$$|\log L(s,\chi)| \leq \log\log q\tau + O(1),$$

$$\frac{1}{\log q\tau} \ll |L(s,\chi)| \ll \log q\tau.$$

- If $|s-\beta_1| \leq \frac{1}{\log q}$, then

$$\frac{L'}{L}(s,\chi) = \frac{1}{s-\beta_1} + O(\log q\tau)$$

$$|\log L(s,\chi)| \leq \log\log q\tau$$

and

$$\frac{1}{\log q\tau} \ll \frac{|L(s,\chi)|}{|s-\beta_1|} \ll \log q\tau.$$