Monday, February 8

Recall the zero-free region for Dirichlet L-functions: there exists $c>0$ such that $L(s,\chi) \neq 0$ in the region $\{\sigma > 1 - \frac{c}{\log q\tau}\}$, except possibly a single real zero β_1 if χ is quadratic.

Theorem 11.7: If $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$ are quadratic characters, not induced by the same primitive character χ^*, then $L(s,\chi_1)L(s,\chi_2)$ has at most one (real) zero in $\{\sigma > 1 - \frac{c}{\log q_1 q_2 \tau}\}$.

Note: if $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$ are both induced by $\chi^* \pmod{q^*}$, then

$$L(s,\chi_1)\prod_{\substack{p \mid q_1 \\ p \nmid q^*}}\left(1-\frac{\chi_2(p)}{p^s}\right) = L(s,\chi^*) = L(s,\chi_2)\prod_{\substack{p \mid q_2 \\ p \nmid q^*}}\left(1-\frac{\chi_2(p)}{p^s}\right)$$

So if $L(s,\chi_1)$ has an exceptional zero, then $L(s,\chi_2)$ has the same zero.

Proof: For $\sigma > 1$,

$$-\frac{\zeta'}{\zeta}(\sigma) - \frac{L'}{L}(\sigma,\chi_1) - \frac{L'}{L}(\sigma,\chi_2) - \frac{L'}{L}(\sigma,\chi_1\chi_2)$$

$$= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma}\left(1 + \chi_1(n) + \chi_2(n) + \chi_1\chi_2(n)\right)$$

$$= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma}(1+\chi_1(n))(1+\chi_2(n)) \geq 0.$$

- $-\frac{\zeta'}{\zeta}(\sigma)$ has a pole at $\sigma = 1$
- $\chi_1\chi_2$ is nonprincipal, so $-\frac{L'}{L}(\sigma,\chi_1\chi_2)$ has no pole.
- If $L(s,\chi_1)$ and $L(s,\chi_2)$ had zeros, this would violate nonnegativity ("two vs. one").

Corollary 11.10: For every $Q > 1$,

$$\prod_{q \le Q} \prod_{\substack{\chi \,(\mathrm{mod}\, q) \\ \chi \text{ primitive}}} L(s,\chi) \quad \text{has no zeros in } \left\{ \sigma > 1 - \frac{c}{\log Q\tau} \right\},$$

except possibly one real zero (from a quadratic character).

Corollary 11.9: For every $A > 0$, there exists $c(A)$ such that if $q_1 < q_2$, and $L(\beta_1, \chi_1) = 0$ with

$$\beta_1 > 1 - \frac{c(A)}{\log q_1},$$

and $L(\beta_2, \chi_2) = 0$ with $\beta_2 > 1 - \frac{c(A)}{\log q_2}$ (where $\chi_1 \,(\mathrm{mod}\, q_1)$ and $\chi_2 \,(\mathrm{mod}\, q_2)$ are primitive)

Then $q_2 > q_1^A$.

Definition: An integer d is a fundamental discriminant if either:

- $d \equiv 1 \pmod 4$ and d is squarefree; or
- $4 \mid d$, $\frac{d}{4} \equiv 2, 3 \pmod 4$, and $\frac{d}{4}$ is squarefree.

Fact:

- bijection between fundamental discriminants and quadratic extensions $\mathbb{Q}(\sqrt{d})$ of $\mathbb{Q}$ (except $\mathbb{Q}(\sqrt{1}) = \mathbb{Q}$)
- bijection between fundamental discriminants and primitive real Dirichlet characters $\chi_d(n) = \left(\frac{d}{n}\right)$, using Kronecker's extension of the Legendre/Jacobi symbol.

$\chi_d(n)$ has conductor $|d|$ and $\chi_d(-1) = \operatorname{sign}(d)$.

Dirichlet's class number formula

Let d be a fundamental discriminant, and let $h(d)$ be the class number of $\mathbb{Q}(\sqrt{d})$.

- If $d<0$, then $L(1,\chi_d) = \frac{2\pi h(d)}{w\sqrt{|d|}}$, where w is the number of roots of unity in $\mathbb{Q}(\sqrt{d})$. (Indeed, $\mathbb{Q}(\sqrt{-1})$ has 4 roots of unity, and $\mathbb{Q}(\sqrt{-3})$ has 6 roots of unity, but otherwise $w=2$.)

Remark: we computed earlier

$L(1,\chi_{-3}) = \frac{\pi}{3\sqrt{3}}$, $L(1,\chi_{-4}) = \frac{\pi}{4}$,

$L(1,\chi_{-7}) = \frac{\pi}{\sqrt{7}}$; this proves that

$h(-3) = h(-4) = h(-7) = 1$.

- If $d>0$, then $L(1,\chi_d) = \frac{h(d)\log\eta}{\sqrt{d}}$, where $\eta = \frac{1}{2}\log(x_0 + y_0\sqrt{d})$, where (x_0,y_0) is the smallest positive solution to $x^2 - dy^2 = 4$.
  - η is either a fundamental unit or the square of a fundamental unit.

Corollary: $L(1,\chi_d) \gg \frac{1}{\sqrt{|d|}}$.

Recall: If β_1 is an exceptional zero of $L(s,\chi)$, then for s near 1,

$$\underset{(\chi \text{ primitive})}{\frac{1}{\log q\tau}} \ll \left| \frac{L(s,\chi)}{(s-\beta_1)\log q} \right| \ll \log q\tau.$$

Consequently: Corollary 11.12

There exists $c>0$ such that

$\left\{\sigma > 1 - \frac{c}{q^{1/2}\log^2 q\tau}\right\}$ really is zero-free

Theorem 11.14 (Siegel): For every $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that if $\chi \pmod q$ is quadratic, then

$$L(1,\chi) > C(\varepsilon) q^{-\varepsilon}.$$

$$L(1,\chi) \gg_\varepsilon q^{-\varepsilon}.$$

Corollary 11.15: If β_1 is an exceptional zero, then $1-\beta_1 \gg_\varepsilon q^{-\varepsilon}$.

Remark: The proof of Theorem 11.14 is "ineffective": there's no way to work out the value of $C(\varepsilon)$, even in principle.

- proof of Theorem 11.14 depends on whether any exceptional zeros exist.

Back to $\pi(x;q,a) = \sum_{\substack{p \leq x \\ p \equiv a \pmod q}} 1$

and $\psi(x;q,a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} \Lambda(n)$

Since $\frac{1}{\phi(q)} \sum_{\chi \pmod q} \bar{\chi}(a)\chi(n) = \begin{cases} 1, & \text{if } n \equiv a \pmod q \\ 0, & \text{otherwise} \end{cases}$

(when $(a,q)=1$),

we have for $(a,q)=1$,

$$\psi(x;q,a) = \sum_{n \leq x} \Lambda(n) \frac{1}{\phi(q)} \sum_{\chi \pmod q} \bar{\chi}(a)\chi(n)$$

$$= \frac{1}{\phi(q)} \sum_{\chi \pmod q} \bar{\chi}(a) \psi(x,\chi) \text{ where}$$

$$\psi(x,\chi) = \sum_{n \leq x} \Lambda(n)\chi(n).$$

Perron's formula:

$$\psi_0(x,\chi) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} -\frac{L'}{L}(s,\chi) \frac{x^s}{s} ds,$$

$$\psi_0(x,\chi) = \frac{1}{2}(\psi(x^-,\chi), \psi(x^+,\chi)).$$

Theorem 12.10: Fix $c>1$. For $x \geq c$, $T \geq 2$: if $\chi \pmod q$ is primitive then

$$\psi_0(x,\chi) = \left\{\frac{x}{\phi(q)} \text{ if } q=1\right\} - \sum_{\substack{\rho \\ |\gamma| \leq T}} \frac{x^\rho}{\rho} - \frac{1}{2}\log(x-1)$$

$$- \frac{\chi(-1)}{2}\log(x+1) + \left\{\frac{L'}{L}(1,\bar{\chi}) \text{ if } q>1\right\}$$

$$+ \log\frac{q}{2\pi} - C_0 \text{ (Euler's constant)}$$

$$+ R(x,T,\chi)$$

where $R(x,T,\chi) \ll \log x \cdot \min\left\{1, \frac{x}{T\langle x\rangle}\right\}$

$$+ \frac{x}{T}(\log(qxT))^2.$$

- $\langle x\rangle$ = distance from x to nearest prime power not counting x itself

- $\sum_\rho$ is a sum over nontrivial zeros of $L(s,\chi)$ (so $0<\beta<1$).

Letting $T \to \infty$:

$$\psi_0(x,\chi) = \left\{\frac{x}{\phi(q)} \text{ if } q=1\right\} - \sum_\rho \frac{x^\rho}{\rho} + \text{some other terms.}$$

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"explicit formula"