Prime counting functions: Monday, January 11

$$\pi(x) = \#\{\text{primes} \le x\} = \sum_{p \le x} 1$$

$$\theta(x) = \sum_{p \le x} \log p$$

$$\psi(x) = \sum_{n \le x} \Lambda(n), \text{ where}$$

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^r, \\ 0, & \text{otherwise.} \end{cases}$$

- $\theta(x)$ is roughly $\pi(x) \log x$.
- $\psi(x) = \theta(x) + \theta(x^{\frac{1}{2}}) + \theta(x^{\frac{1}{3}}) + \cdots$

$$= \theta(x) + \theta(x^{\frac{1}{2}}) + \text{negligible}$$

Prime number theorem:

☆ $\psi(x) \sim x$ (meaning $\lim_{x \to \infty} \frac{\psi(x)}{x} = 1$)

☆ $\theta(x) \sim x$

$\pi(x) \sim \frac{x}{\log x}$ (natural logarithm)

Define $Li(x) = \int_2^x \frac{1}{\log t}\, dt.$

$Li(x) \sim \frac{x}{\log x}$; in fact,

$$Li(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \cdots$$

$\pi(x) \sim Li(x).$

But:

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)$$

$$\pi(x) = Li(x) + O\left(x \exp(-c\sqrt{\log x})\right)$$

☆ for some absolute constant $c > 0$.

$$\Rightarrow \pi(x) = Li(x) + O_A\left(\frac{x}{\log^A x}\right)$$

Assuming Riemann hypothesis:

$$\psi(x) = x + O(\sqrt{x} \log^2 x)$$

$$\theta(x) = x + O(\sqrt{x} \log^2 x)$$

$$\pi(x) = Li(x) + O(\sqrt{x} \log x).$$

Riemann zeta function: function of complex variable $s = \sigma + it$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \text{ converges (absolutely) when } \mathrm{Re}\, s > 1 \quad (\sigma > 1)$$

Euler product, converges absolutely for $\sigma > 1$:

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1}.$$

$\Rightarrow \zeta(s) \neq 0$ for $\sigma > 1$.

$\zeta(s)$ extends via analytic continuation to $\mathbb{C} \setminus \{1\}$. $\zeta(s)$ has a simple pole at $s=1$, with residue 1.

Functional equation. Define

$$\xi(s) = \frac{1}{2} s(s-1)\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}.$$

Then $\xi(s)$ is entire, and

$$\xi(s) = \xi(1-s).$$

Zero free region:

- $\zeta(s) \neq 0$ when $\sigma = 1$
  $\iff \pi(x) \sim \mathrm{Li}(x)$
- $\zeta(s) \neq 0$ when $\sigma > 1 - \frac{c}{\log t}$.